Anatomy of a Shale Boom: Optimal Leasing and Drilling with Costly Search

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Abstract

U.S. shale plays tend to first see an initial land rush as firms lease minerals, followed by a long delay before drilling picks up. Based on the characteristics of the mineral leasing process and descriptive statistics from South Texas' Eagle Ford shale, I argue that this is due to search frictions in the market for mineral rights. I construct a dynamic, general equilibrium model of firms' joint leasing and drilling decisions when costly search for leases is required, and I characterize the equilibrium path of leasing and drilling using continuous time optimal control methods. The model shows that along the optimal path of leasing and drilling, firms accelerate leasing activity to avoid high search costs when unleased acreage becomes scarce. This dynamic does not arise in a frictionless market unless there is uncertainty in price. In addition to leasing, I also include technological change and a capital-intensive oilfield services sector. With the addition of these two features, the model can explain the qualitative dynamics of shale development in South Texas' Eagle Ford shale.

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1 Introduction

Private ownership of mineral rights is a unique feature of the United States oil and gas market. Since the beginning of the shale-boom in the mid-2000s, mineral royalties have been a substantial source of income for these private owners. (Brown, Fitzgerald, and Weber, 2016) estimate that owners in the largest plays collectively received $39 billion dollars in royalty revenues from their minerals, dwarfing farm transfers.

The market for privately owned mineral rights, however, is not well understood. There are a large number of papers on auctions for mineral rights—the typical market mechanism used by governments to lease mineral rights—but private transactions are usually done through bilateral negotiations, not formal auctions. The depletable resource literature usually ignores the process of transferring mineral rights and skips ahead to situations in which the owner and exploiter of a resource are the same. These two literatures provide little guidance on what the drivers of a private market for mineral rights are, how we should expect the market to evolve over the life of a shale play, or how it should respond to unexpected changes in price or technology.

The recent increse in extraction of shale gas and light tight oil is often referred to as a "boom" because it has been unusually rapid and large in magnitude. First, an initial land-grab happens in which companies lease up large tracts of land. Then there is a substantial delay before leases are drilled. I argue that the market for privately-owned mineral rights can be characterized by decentralized, costly search. I explicitly link the value of mineral leases to firms' optimal extraction problem and show how search costs can match the qualitative features of the leasing and drilling dynamics that we have seen empirically. The model focuses on the transition dynamics of a large initial stock of unleased mineral rights to a very small ending stock of unleased mineral rights, reflecting what we have seen empirically.

Explaining this delay using a decentralized market in transition contrasts sharply with a frictionless, centralized market. In such a world, delay will only occur when prices or costs are random. With decentralized search in a non-stationary environment, it can be optimal for competitive firms to accelerate their lease purchases ahead of their drilling investments because initial search costs are low when the supply of unleased mineral rights is large. I find that forward-looking landowners who lease at the beginning of the boom should capture a greater share of expected resource rents compared to landowners who lease later. Prices can rise initially in current-value terms as unleased mineral rights become scarcer, but they decrease at the end of the the leasing cycle since the last landowner to be leased is difficult to locate and firms are reluctant to incur large search costs for marginal benefits.

In addition to the private ownership of mineral rights by many individuals, the North American shale boom has also been characterized by a robust oilfield services sector, learning, low barriers to entry, and operator heterogeneity (Medlock, 2014a; Medlock, 2014b). Since the value of mineral leases is derived from the price of the underlying resource and the extraction process, I study how these four factors have affected both the pace and price of leasing minerals through their effect on the path of extraction. In general, factors which lower the cost of extraction or increase recovery rates increase the value of leases directly by increasing the profitability of drilling and indirectly by speeding up the time to depletion.

In the next section (Section 2), I further describe how the market for private mineral rights works and use one example of a shale boom—South Texas' Eagle Ford Shale to illustrate the most important dynamics of the boom. Section 3 is a brief literature review. Motivated by the qualitative features of the dynamics of a shale boom presented, I set up an economic model in Section 4. Even without any further results, this allows for important insights into landowners' value of owning a mineral lease. Section 5 proceeds to define and characterize the equilibrium using optimal control theory. In Section 6, I make more specific assumptions about particular functional forms to prove several results about the equilibrium. Finally, I discuss numerical simulations of the model that illustrate how the model qualitatively replicates the important dynamics of the Eagle Ford shale boom and recent downturn.

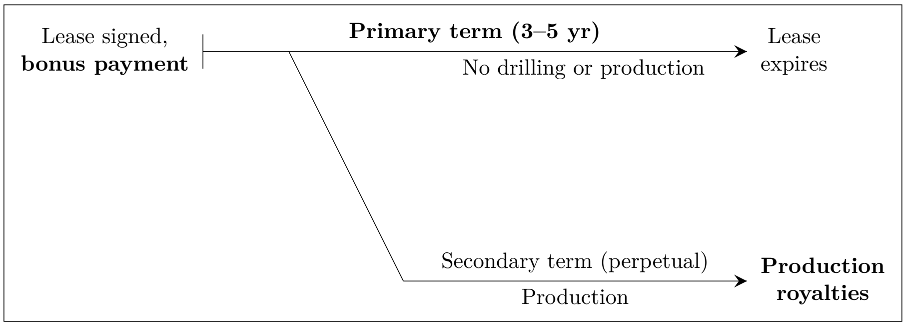
2 Institutional details and the Eagle Ford

2.1 Leasing

In most countries other than the United States, sub-surface oil and gas resources are owned by the State; however, in the US, property rights include sub-surface minerals. This has meant private individuals have owned much of the right to develop shale resources. Before a company can extract minerals, it must purchase or rights to do so from the owners, usually by executing a mineral lease.[1] Figure 1 shows a diagram of a mineral lease contract. Leases

[1]An alternative is for companies to grant mineral owners a working interest in the well, meaning that the owner shares in the costs of developement as well as revenues.

Figure 1: Structure of a mineral lease



generally require an up-front cash payment (the bonus bid) plus a percentage of revenues from extracted minerals (the royalty rate). Leases are usually not perpetual; instead, they specify an initial primary term, usually from three to five years. Activity must commence by the expiration of the primary term, or the lease is forfeit. Once a lease has been drilled, as long as production continues in commercial quantities, the lease enters the secondary term, is considered to be held by production, and can be retained by the lessee (purchaser) indefinitely.

With government-owned minerals, leases are usually sold through an auction process. Though a small fraction of minerals in U.S. shales are government-owned and sold in this manner, most are not. Instead, companies purchase minerals through bilateral negotiations with landowners. This decentralized process may present lower barriers to entry for less sophisticated or smaller operators, increasing market entry and competition. Because US landowners are direct financial beneficiaries of extraction through royalty payments, local political resistance to drilling is likely lower compared to other countries, as has been the case in Europe, which has seen strong opposition to shale development.

Like the labor-market, the process of leasing privately-owned mineral rights is characterized by costly search and bilateral negotiations over a heterogeneous good. No two leases are alike since they are by definition located in different places. Geology, the distance to other leases in a firm's portfolio, and distance to gathering and processing infrastructure all change over space. After identifying an area of interest, a firm must examine historical records, often housed at the local county courthouse, to determine whether each tract of land has already been leased, and trace ownership of the mineral rights. Since mineral rights can be severed from surface rights, the current surface owners often do not own the sub-surface minerals. The original deed can date back many decades, and the current mineral owners may live in an entirely different state. Locating and negotiating with landowners can be especially costly if mineral interests have been split among many family members.

During a lease negotiation, the opportunity cost for a landowner is the next offer she would receive, plus any disruption drilling might impose on her. This cost is her threat-point in a negotiation and will influence the terms of the lease. Offers arrive randomly and infrequently, so the threat-point should evolve with expectations about the tightness of the mineral rights market and the prevailing lease terms in the future. During a leasing boom, the supply of unleased acreage can decrease rapidly and permanently. Analyzing the dynamics of a boom cannot be done by analyzing a steady-state, rather one must look at non-stationary transition dynamics.

I choose to model the amount of lease-able mineral acres as a finite quantity, just as models of physical depletion of a resource in the vein of Hotelling (1931) postulate a finite quantity of the resource. This is because I am modeling a specific shale play in a specific geographic area. Though the economically viable quantity of shale resources does expand within a finite geographic boundary as the resource price increases and technology improves, the quantity of sub-surface rights within the same area does not. Assuming a finite quantity of mineral acres allows us to model the dramatic increase in scarcity of unleased acreage that we see empirically and focus on transition dynamics of a land-rush. Though allowing the stock mineral rights to expand may be a better reflection of the long-run evolution of shale resources, a finite quantity better reflects the shorter-run transition this paper is interested in.

One good example of such a "land-grab" curred in South Texas' Eagle Ford Shale (shown in Figure 2). The Eagle Ford covers an area of around 33,000 square kilometers, about the size of Taiwan or Belgium.[2] It contains three windows of different thermal maturity (in increasing order): oil, wet or rich gas, and dry gas. Though leasing, drilling, and production began in the Eagle Ford well before the 21st century shale boom, earlier activity had no where near the same scale or intensity of the activity that started in 2005. More than 50% of its area was leased during the three years 2008–2011 as shown in Figure 3a. As of May 2016, more than 60% of the total area in the Eagle Ford had been leased in the more than 20,000 separate transactions digitized by Drillinginfo. Figure 3b maps the tracts leased in the Eagle Ford by year and illustrates how densely leases cover

[2] I base my definition of the Eagle Ford's geographic extent and division on digital maps from Drillinginfo.

Figure 2: The Eagle Ford shale

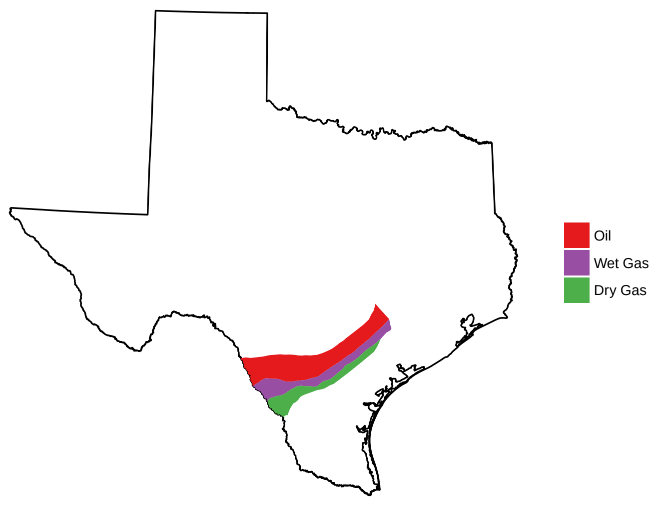
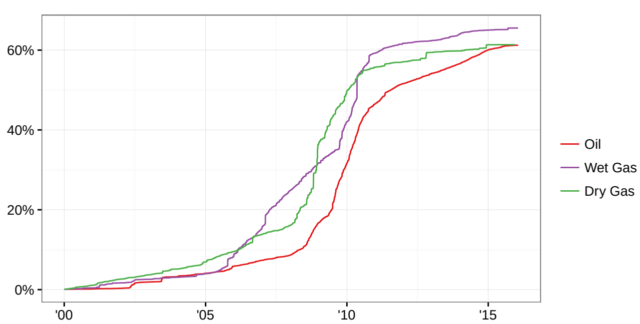


Figure 3: Leasing in the Eagle Ford

(a) Share of surface area leased



(b) Map of leases by year

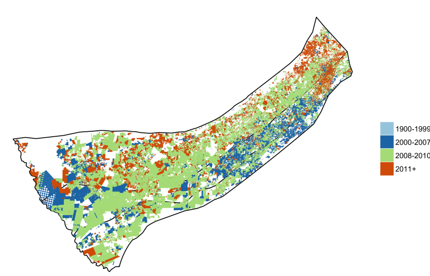
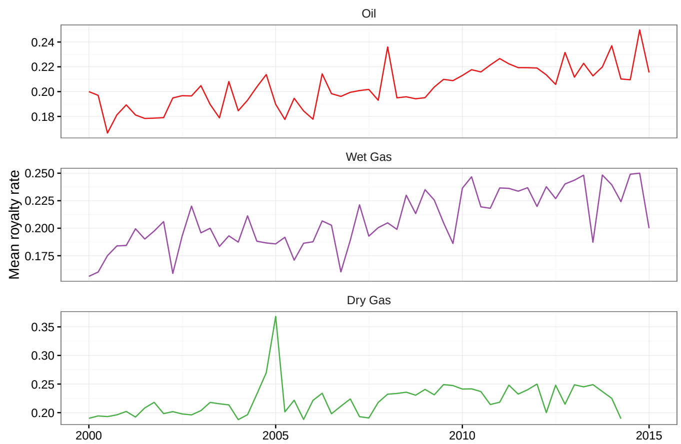


Figure 4: Mean royalty rates



the shale. Figure 4 shows how mean royalty rates rose as the supply of unleased acres fell (oil prices also rose significantly during this period).[3] As the plots make clear, the shale boom has been a period of transition—not a steady state, so understanding this market requires understanding the transition dynamics of the market.

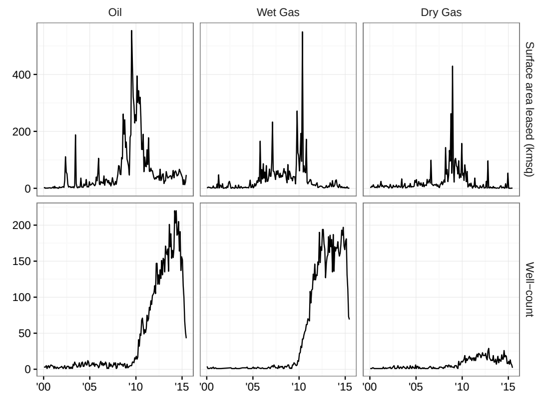
2.2 Timing: drilling vs leasing

After acquiring a mineral lease, firms tend to delay drilling the associated location. Figure 5a shows that in aggregate, most leasing in the Eagle Ford pre-dated the run-up in drilling rates by a few years. Figure 5b shows estimates of the drilling hazard rate for for three year Eagle Ford leases that did and did not include options to extend the primary term by two years. The large rise in the probability of drilling right before three and five years suggests that operators delay drilling as much as possible but are constrained by the primary term. Purchasing a lease only to wait years to drill is costly, and there are two explanations for this behavior: real options and search costs during a period of transition. Real options theory suggests that firms delay drilling because in a world of uncertain prices, preserving the option of drilling at a possibly higher price is valuable. This paper shows that when firms know search costs will increase in the future as the market for mineral rights tightens, it can be optimal to accelerate leasing to take advantage of the present low search costs. While this explanation does not require stochastic prices, it does require that

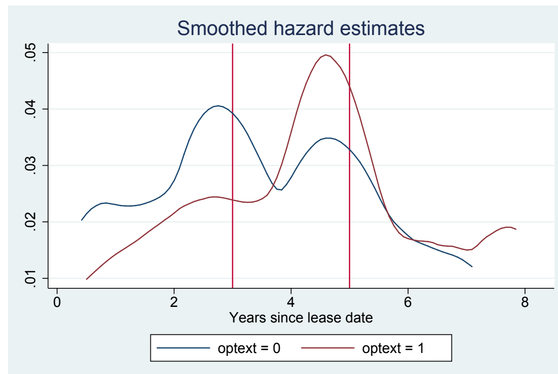
[3] Mean royalty rates are calculated as an area-weighted average.

Figure 5: Firms delayed drilling in the Eagle Ford

(a) Monthly leasing and drilling rates



(b) Hazard rates for drilling 3 year leases



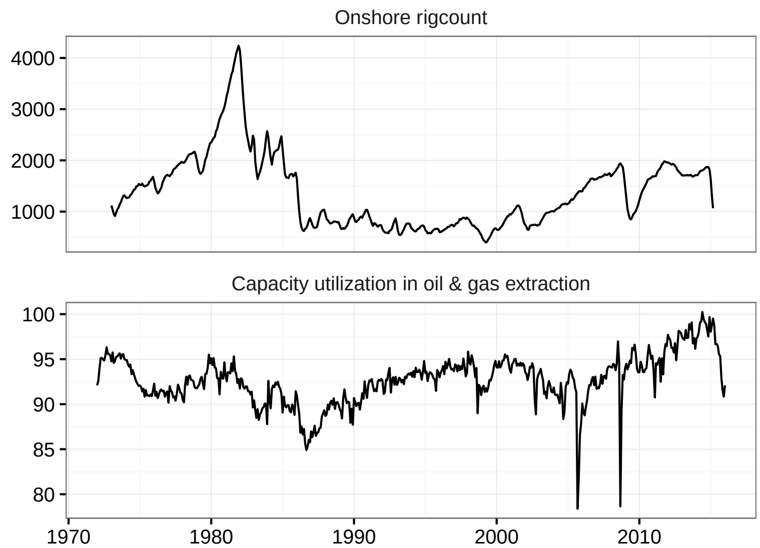
the leasing market be in a transition period, not a steady-state. The oil and gas industry ran at full capacity for much of the shale boom, suggesting that operators were drilling wells as fast as they could. Figure 6 shows that capacity utilization from the Federal Reserve Bank's industrial production statistics has been very high in the upstream sector. In fact, in June 2014 at the peak of the boom, it was over 100%.

2.3 Learning

Extracting oil and gas from shale was made possible by technological innovation, and companies have continued to learn as they drill more. Figure 7 how initial production (IP) rates in Eagle Ford of wells have climbed dramatically over the shale boom.[4] Though some of the increases in IP rates is due to the fact that firms drilled longer wells and learned where geologically productive "sweet spots" are, some of the productivity gains are also due to learning. This learning may be either private learning-by-doing or may spread through industry-wide spillovers. Covert (2015), Kellogg (2011), and Seitlheko (2015) all find that learning appears to be company-specific. Nevertheless, one might expect some knowledge spillovers to be present since the technologies developed by shared oilfield services provideers are available to all operators. One early shale entrepreneur—George Mitchell—realized that by combining the already understood techniques of hydraulic fracturing and

[4]Computed from Drillinginfo well-header data as a 50-well moving average of the sum of peak gas divided by six plus peak oil.

Figure 6: US onshore rigcount and capacity utilization (Jan 1972–Jan 2016)

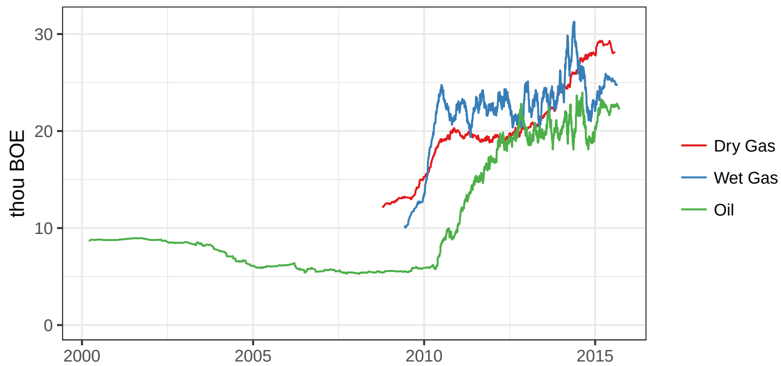


horizontal drilling, one could profitably extract oil and gas from hitherto uneconomic formations. The innovation spread quickly throughout the industry

Technological progress and learning have important implications for both landowners and operators. Rational landowners should account for future productivity gains when negotiating leases. Firms should account for technological progress as well, both when buying the lease and drilling it. Furthermore, we should also expect private learning-by-doing and knowledge spillovers to have distinct effects, even when costs are identical. Private learning should increase the payoff from drilling to a firm since it captures the gains from learning; this should increase the value of leases and the pace of drilling, because it decreases the opportunity cost of extraction and makes lower initial cash-flows optimal. Because they not internalized, the presence of knowledge spillovers should slow drilling, at least compared to the case of private learning, since firms would prefer to free-ride off of other companies' experience. Though this will make leases more valuable at the time they are drilled, because drilling will be delayed, it may actually increase or decrease their value. The model I set up can easily accommodate both types of productivyt improvements, and I verify the above intuition mathematically.[5]

[5] I choose to parameterize learning so that it reduces input costs instead of increasing output; however,

Figure 7: Initial production per well



2.4 Free entry and firm heterogeneity

As mentioned earlier, there are relatively low barriers to entry to shale. Though searching for leases and conducting many bilateral negotiations is costly, there are few large fixed costs that must be paid to enter the market for mineral rights.[6] Shale wells are also relatively inexpensive compared to deepwater and other unconventional resources, and much of the drilling is done by third parties—oilfield services companies. This means that small companies do not need to purchase their own drilling rigs and pumping trucks, costly capital outlays, and need not worry about achieving a large scale to spread out fixed costs. Table 1 shows the number of firms in the Eagle Ford which hold a particular percentage of the leases in the play and the share of wells operated.[7] The large number of small firms suggests that the market is fairly competitive, though there are a few large firms.

Table 1 also reflects wide cross-sectional variation in firm sizes in shale. The model

changing this is straightforward. Random learning or Bayesian learning about productivity would add a considerable complication since the ordinary differential equations would all become partial differential equations.

[6] In fact, there there is a profession that specializes in leasing minerals—landmen. These individuals have specialized training, usually much of it legal in nature, in finding, determining ownership, and negotiating leases. Most landmen work for small, independent companies—a telltale sign of a market with low barriers to entry.

[7] Acreage share is calculated from Drillinginfo's "leasing analytics" dataset which attempts to link leases to the final company that owns them (not the middleman which may have acquired the lease on the firm's behalf). This likely over-estiamtes the number of operators who hold acreage positions since not all of the acreage can be assigned. Share of wells operated tabulates the number of wells per "current operator" in Drillinginfo's "well-header" dataset. This likely under-estimates the number of firms with active interests in the Eagle Ford since multiple exploratino and development firms may participate in a well, but only one is the operator.

Table 1: Share of leased acreage and wells operated

Share	Acreage holders	Well operators
6.5–1%	0	3
5–6.49%	2	2
2–4.99%	9	9
1–1.99%	13	9
<1%	1365	432

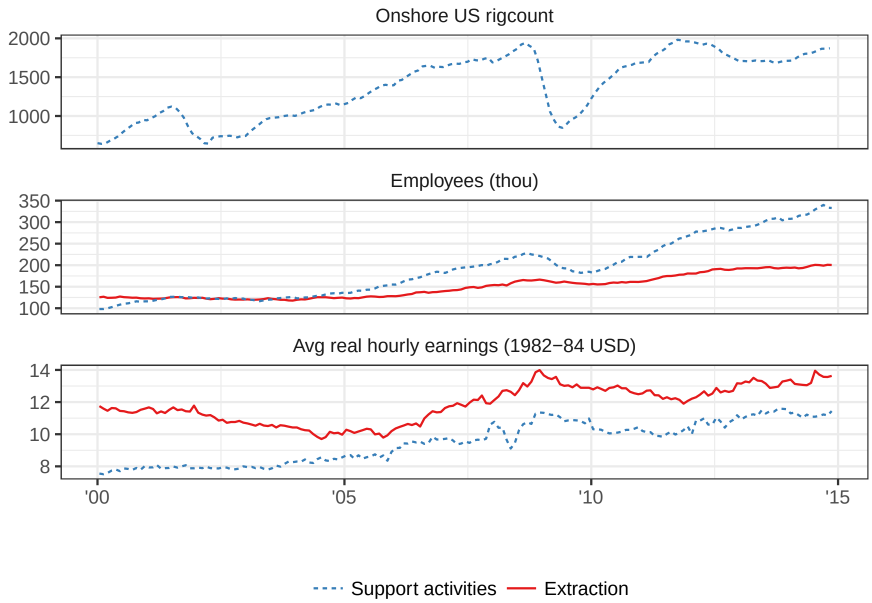
presented in this paper explains such variation by assuming that companies are vary in their productivity, either getting more out of each well, or being able to drill wells in a less costly way. For example, EOG is known as a very efficient operator which is able to coax more than most operators can from its wells and can drill at a lower cost than other firms. Applying stochastic frontier analysis to data on well completion practices and production stochastic frontier analysis, Seitlheko (2015) finds that operators in the Barnett shale exhibit meaningful differences in productivity.

I assume, like Acemoglu and Hawkins (2014) do, that firms are the same before entering the market, and learn their productivity types once they pay a fixed cost of entry. Because the payoff to drilling is different each type values mineral leases differently. Firms that value leases more because they are more productive purchase more leases and, hence, drill more. Thus, the model generates an endogenous cross-sectional distribution of mineral lease portfolios and drilling histories. As the cost of entry drops, the number of firms that choose to enter should increase. Intuitively, a larger number of firms should lead to more competition for leasing and faster drilling.

2.5 Oilfield services

The final enabler of shale has been the oilfield services industry. Operators which own mineral lease positions do not generally drill the wells themselves. Instead, they contract with an oilfield service provider to do so. Oilfield service providers maintain their own labor force, and they own the rigs, pumping trucks, and other capital equipment required to drill and complete wells. There are four large, well-known firms—Schlumberger, Halliburton, Baker-Hughes, and Weatherford—but also a number of smaller companies. The ability of the oilfield service sector to grow rapidly by increasing its labor-force and capital stock has allowed the oil and gas industry to rapidly increase drilling and production from

Figure 8: Oilfield services enabled growth



shale. During the shale boom, historically low interest rates have allowed firms to increase investment. This paper's model shows how low capital costs translate into low drilling costs, higher profits for operators, and higher lease prices.

Figure 8 depicts how the dramatic growth of shale drilling generated a much sharper increase in employment by oilfield services (the solid blue line) than the operators (dotted red line).[8] Conversely, as Figure 9 shows, the drop in drilling has led to an equally dramatic fall in rig-counts and employment in services, but the fall in extraction employment has been much lower.

Figure 9 shows how costs, rig-counts, employment, and wages responded to the oil price crash of 2014.[9] OPEC's November 2014 decision to sustain production, denoted by the vertical line, started a sharp decline in shale investment. Again, a highly responsive oilfield services sector was able to quickly idle rigs and pare back its workforce. Like

[8]Employment and wage data are from the Bureau of Labor Statistics (BLS) and correspond to NAICS code 211 (oil and gas extraction) and NAICS 213112 (oilfield services). Onshore U.S. rig-counts are from Baker Hughes.

[9]WTI is taken from the EIA's website. The PPI is taken from the BLS via the FRED website.

employment, costs rose quickly as demand for oilfield services increased,[10] and it also fell quickly as the oil price and demand for oilfield services did. Many have cited this drop in costs as a reason that shale investment remained as high as it did after the 2014 drop in oil prices. Interestingly, the recent drop in price and activity coincided with a sharp rise in wages for the extraction industry. The counter-cyclical movement in extraction wages indicates increasing labor productivity. This would be consistent with an industry which has focused on its most productive wells and terminated employment for less productive employees while retaining senior, more productive ones. In contrast, the fall in oilfield services employment has coincided with a very slight fall in wages.

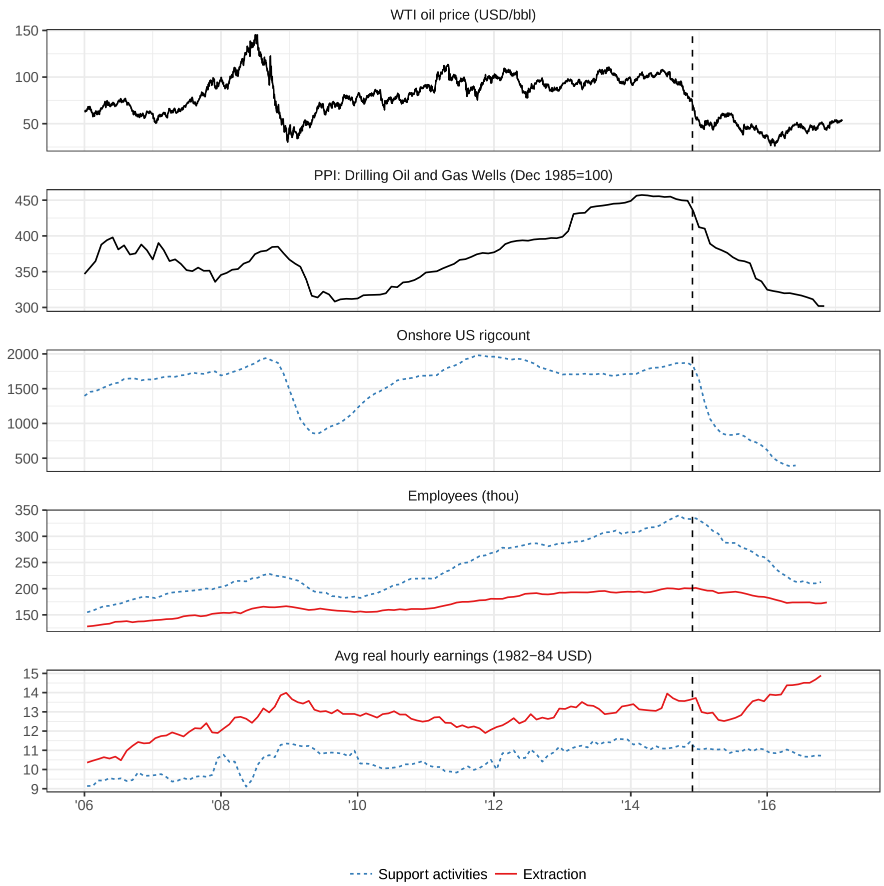
By separating the extraction and oilfield services sectors, the model presented in this paper can mimic the qualitative response of employment and costs to an unexpected price drop. Drilling rates drop quickly in response to a fall price, but can be sustained at somewhat higher levels than they otherwise would have been because the oilfield services sector absorbs some of this fall in price.

3 Literature review

Empirical research on the market for private mineral rights is still fairly limited and generally treats it in a static framework. Brown, Fitzgerald, and Weber (2016) estimate how royalty rates vary with geological quality of leases, and Brown et al. (2016) study how royalty revenues are distributed geographically—whether they are paid to local owners or absentee owners in other counties or states. Brown, Fitzgerald, and Weber (2016) does elaborate a simple, centralized model of mineral rights in which an upward-sloping supply curve is derived from a cross-sectional distribution of landowners' opportunity costs (presumably the disruption from drilling). The authors explain that they choose not to focus on the dynamics of leasing since leasing has been rapid, suppressing meaningful dynamics. Royalty rates in some plays have a clear, upward trend over time, and landowners' option to walk away from negotiations and sell to the next firm is, by its very nature, a dynamic phenomenon. This suggests that dynamics are, in fact, important. Unfortunately, when taken to a dynamic setting in which the stock of available acreage is leased up over time and leasing activity has a non-zero up-front cost, the model in Brown, Fitzgerald, and Weber (2016) makes two strange predictions. First, the marginal landowner's opportunity

[10] Producers started drilling the longer, more expensive horizontal wells, which is partially reflected in the upward trend.

Figure 9: Price, costs, rig-counts, employment, and wages



cost of leasing ratchets up as leases are acquired by firms. While that is not problematic in and of itself, it would also imply that a fall in the price of oil or gas, if not accompanied by an exactly offsetting reduction in extraction costs, would cause an *increase* in the royalty rate—the opposite of what one would expect. Second, because firms do incur a cost to lease acreage, they can raise profits by delaying leasing expenditures as much as possible.[11] That prediction is definitely not borne out in the data: firms tend to delay drilling by a substantial amount. The model I propose addresses both of these issues: landowners' opportunity cost of leasing is endogenously determined based on their expectations of future supply and demand in the leasing market, and the presence of search-costs naturally causes firms to accelerate their leasing compared to their drilling. The cost of adding these features is added complexity.

Timmins and Vissing (2014) and Vissing (2015, 2016) are three more purely empirical studies that examine the non-pecuniary terms of mineral rights in Texas' Barnett shale. Using data on these, they find that owners with higher socio-economic status are able to extract higher surpluses. This is evidenced by the increased number of costly restrictions on drilling that favor the landowner. They also find that firms with geographic concentrations of leases have higher bargaining power and higher valuations for the leases. As I do, Vissing (2016) treats leases as the outcome of a one-to-many matching process and remains agnostic about the exact bargaining process. Like the previous papers mentioned, however, the framework is a static, partial equilibrium model with heterogeneneous royalty owners and firms. My framework is a dynamic, general equilibrium with limited firm heterogeneity.

Since Hotelling (1931), many papers have considered the problem of optimal extraction of a depletable resource in the aggregate. Most recently, Anderson, Kellogg, and Salant (2014) adapt the Hotelling framework to better fit the situation of the oil and gas industry. The authors' model uses geological constraints on well flow to rationalize why producers do not choke back production when prices drop, a prediction that previous models had not been able to match. Despite the many papers written on optimal extraction of a depletable resource, most assume that the owner and exploiter of the resource are the same; none model the initial transfer of exploration and production rights that must occur before exploitation. This paper extends this literature by incorporating the initial transfer of mineral rights into a depletable resource framework.

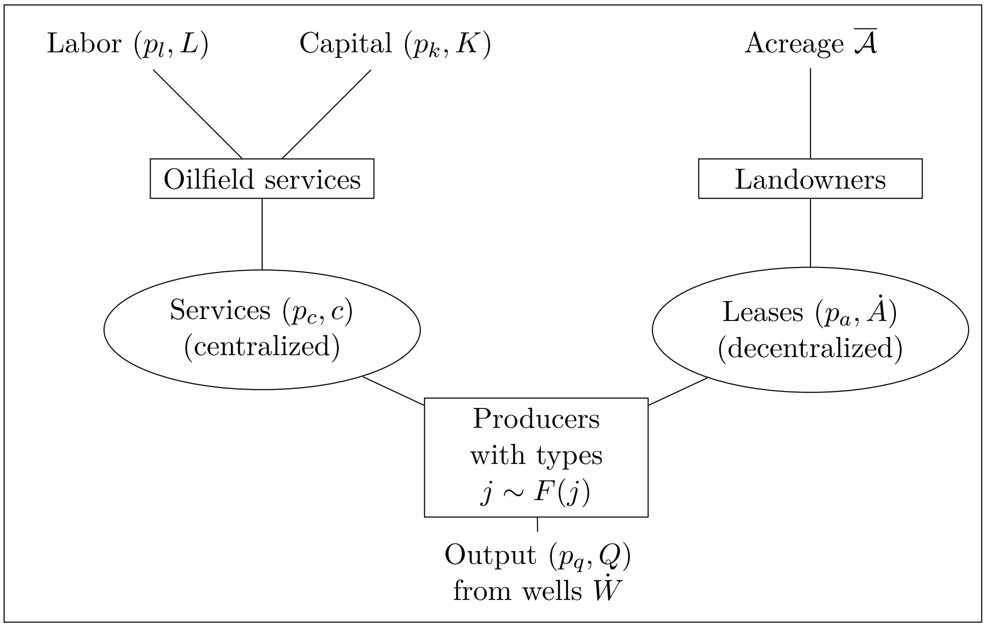
[11] While Brown, Fitzgerald, and Weber (2016) do not include an up-front cost for leasing, they do acknowledge that most leases specify both a royalty rate as well as an up-front bonus to be paid to the landowner.

An empirical regularity is that firms tend to delay drilling wells on their leases. This is true in both the Eagle Ford shale, as I describe later, as well as in offshore wells (Hendricks and Porter, 1996). One explanation for this is that a mineral lease is a real option. In a world with random prices, the *option* to drill has value, and firms wait to exercise it (drill the lease). Paddock, Siegel, and Smith (1988) study mineral leases as a real option from a theoretical point of view. Recently, Smith (2014) and Smith and Thompson (2008) show how to adapt the real options framework to situations when geology is uncertain and correlated, and when leases expire within three years unless a well is drilled. Kellogg (2014) takes real options theory to data on drilling and finds that firms do, in fact, treat leases as a real option. The alternative explanation presented in this paper is not that firms delay drilling to preserve leases' option values but that they accelerate leasing because low initial search costs will rise in the future. This leads to a gap between when leases are purchased and drilled. This alternative theory may fit better with the fact that the oil and gas industry as a whole ran at full capacity for much of the shale boom. Figure 6 shows that capacity utilization from the Federal Reserve Bank's industrial production statistics has been been very high in the upstream sector for the past several years. In fact, in June 2014, it was over 100%.

A series of seminal papers by Hendricks and Porter on government auctions of offshore mineral leases (Hendricks and Kovenock, 1989; Hendricks and Porter, 1993, 1996; Hendricks, Porter, and Boudreau, 1987) do examine investment decisions and the sale of mineral rights to oil companies. However, the market mechanism and institutional details are very different for these government-run offshore auctions. Instead of many smaller firms competing for many small leases owned by many individuals in a shale, the offshore context involves a handful of companies purchasing large tracts from a monopolistic seller. Other empirical papers on oil and gas lease auctions include Kong (2015, 2016), Li, Perrigne, and Vuong (2002), and Porter (1995). Two theoretical papers which study competition between auctions in an equilibrium setting are McAfee (1993) and Peters and Severinov (1997).

Since the pioneering work of Diamond, Mortensen, and Pissarides, random search has been a workhorse of labor economics. The particular flavor of search-model described in this paper is based on Acemoglu and Hawkins (2014) as well as the earlier working-papers Acemoglu and Hawkins (2006) and Acemoglu and Hawkins (2010). One computational obstacle to using search-models in empirical work is that random arrival of matches leads to a distribution of firm sizes. Acemoglu and Hawkins (2014) circumvent this problem by assuming that workers are atomistic compared to firms. Thus, though the identities of

Figure 10: Diagram of market



which workers are matched with which firms are random, the *measure* of workers matched is not. Like a law of large numbers, this turns a stochastic process into a deterministic one. An analog is global hiring by Walmart. Though the identities of workers are random, the firm knows that if it exerts a certain amount of effort on hiring, it will be able to contract a deterministic number of employees. The primary difference between my model and that of Acemoglu and Hawkins (2014) is in the production process. Where they model a simple manufacturing process with only labor inputs, I consider a depletable resource and add an intermediate good (oilfield services). The depletable nature of the resource causes my model to be non-stationary (except for the trivial steady state achieved at depletion), so I limit heterogeneity to be discrete, rather than continuous.

4 Model

In this section, I describe an economic model of a shale boom which is motivated by the institutional characteristics of the market for private mineral rights and the division of the extraction process between oilfield services and operators. Figure 10 diagrams the economic environment. The three economic actors are in rectangular boxes, and the two markets in which they interact are represented by ellipses. Inputs and outputs with their respective prices are represented as ordered pairs. The central agents in the model are the producers of oil and gas (also called operators), which lease mineral rights from atomistic landowners, purchase drilling services from oilfield service companies to extract the resource, and sell the resource into the world market at a constant, exogenous price. Landowners in this model are passive. These infinitely-lived, risk-neutral agents which cannot monetize mineral rights on their own: without a buyer, the rights are worthless. Instead, landowners wait for interested firms to arrive, then engage in negotiations to lease their minerals. I treat oilfield services as a perfectly competitive industry which combines capital and labor to drill wells. Capital (which one could think of as as drilling rigs, pumping trucks, and midstream infrastructure) is long-lived. It is purchased in a competitive market with increasing marginal costs, and capital is accumulated over time. In contrast, labor is purchased in a perfectly elastic market at fixed, exogenous price. Capital stocks adjust more slowly than labor, and they lower the short-run labor costs of drilling. This, in turn, lowers the price of oilfield services.

In the following discussion, calligraphic capital letters denote market-level counterparts of firms' state variables, in capital letters. For example, $\mathcal{A}$ or $\mathcal{W}$ are cumulative aggregate leasing and drilling, and A and W are the firm-level counterparts. Control variables are lower case letters, and firms must optimally choose them. State and control variables are all functions of time, denoted t, and firm-specific variables are also indexed by type, denoted j. To ease notation, I sometimes drop the explicit time and type dependence.

4.1 Matching and negotiation

Mineral rights There is a finite mass $\overline{\mathcal{A}} < \infty$ of homogeneous mineral rights (acres) in the shale. Acres are homogeneous, and the geology is perfectly understood. However, ownership of mineral rights is randomly split between a strictly positive masses of identical non-sellers $\mathcal{A}^n > 0$ and sellers $\mathcal{A}^s > 0$, meaning that $\overline{\mathcal{A}} = \mathcal{A}^n + \mathcal{A}^s$. There are a number of explanations for non-sellers, but a simple one is that drilling imposes a very large disutility on these landowners. The small blank-spaces in the map of leased area in the Eagle Ford (Figure 3b) would be consistent with the presence of such non-sellers. From this point forward, I will use the term "landowner" to mean selling landowners.

A continuum of firms with mass n are indexed by a finite number of types $j \sim F(j)$. At a particular time t, identical firms of type j each own a portfolio of $A(t; j)$ leases. The total area leased at time t is then

$$\mathcal{A}(t) = n \int A(t;j) dF(j) \forall t. \tag{1}$$

The relevant state variable for the firm is not aggregate quantity of leased acres, but the quantity of unleased acres, defined as:

$$\mathcal{U}(t) \equiv \overline{\mathcal{A}} - \mathcal{A}(t). \tag{2}$$

The quantity of acres that have been leased at any time t must satisfy $\mathcal{A}(t) \leq \overline{\mathcal{A}} - \mathcal{A}^n$. Firms cannot distinguish between the two types of landowners during search, so random matching implies that the probability of matching with a type is proportional to the share of land owned by that type. The fact that non-sellers never exit the market means that $\mathcal{U}(t) \geq \mathcal{A}^n > 0 \forall t$ implies that unleased acreage never exceeds total acreage: $\mathcal{U}(t) \leq \overline{\mathcal{A}} < \infty$.The probability that a match will be with a seller and possibly result in a sale is therefore simply

$$\sigma(t) \equiv \frac{\mathcal{U}(t) - \mathcal{A}^n}{\mathcal{U}(t)}. \tag{3}$$

Through σ, the presence of non-sellers causes the matching function to effectively exhibit increasing returns to scale. While unusual in a labor-market context, it is natural for mineral rights. Firms tend to look for larger, contiguous areas to lease, which they assemble into larger portfolios. This becomes increasingly harder to do as the share of unleased acres shrinks. Mathematically, the increasing returns to scale also prevents market tightness from asymptoting to infinity as the mass of unleased acres goes to zero.

Matching At time t, each type of firm j randomly searches for $v(t;j)$ acres, incurring a search cost $\kappa(v)$.[12] Aggregate searching activity simply sums over all firms:

$$\mathcal{V}(t) \equiv n \int v(t;j) dF(j). \tag{4}$$

The aggregate flow of matches is determined by an aggregate matching technology that combines the stock of unleased acres, $\mathcal{U}(t)$, and aggregate searching for unleased acres by

[12] In a labor-search model, v is vacancy posting.

firms, $\mathcal{V}(t)$:

$$M = \begin{cases} M(\mathcal{U},\mathcal{V}) & \text{if } \mathcal{U} > \underline{u} \\ 0 & \text{otherwise} \end{cases}. \tag{5}$$

When unleased acreage reaches a minimum threshold, $\mathcal{U}(t) = \underline{u}$, the flow of matches drops to zero. I assume that $\underline{u} > \mathcal{A}^n$. When unleased acres are above this threshold, the matching function exhibits constant returns to scale in both arguments jointly and decreasing returns to scale in $\mathcal{U}$ and $\mathcal{V}$ separately. This modification ensures that leasing activity stops before drilling does, and it introduces another form of increasing returns to scale to the matching function (as well as a non-convexity). If leasing activity does not stop before drilling does, firms may come to a situation in which they have drilled all of their acreage and immediately drill new leases. Allowing such a situation does not qualitatively affect most of the main results as long as the matching function exhibits (effectively) sufficient increasing returns to scale as $\mathcal{U} \to 0$, and one can make $\underline{u}$ very small.[13] However, setting $\underline{u} = 0$ introduces a binding state constraint which substantially complicates analytical and computational tractability of the model since much of the tractability and insight provided by the model comes from being able to separate the drilling and leasing problems for all periods except the last one. This separation highlights the notion that leases are valuable only insofar as they constrain drilling, and they derive their value at the moment a lack of mineral leases constrains the firm.

Define market tightness in the leasing market as

$$\theta(t) \equiv \frac{\mathcal{V}(t)}{\mathcal{U}(t)}, \tag{6}$$

and assume that the flow of matches generated by a unit of searching as long as $\mathcal{U} > \underline{u}$ can be written as

$$q(\theta) = \frac{M(\mathcal{U},\mathcal{V})}{\mathcal{V}}. \tag{7}$$

Similarly, for both types of landowners with unleased acres (sellers and non-sellers), during

[13] In another, more theoretical paper, I eliminate both the non-sellers and the lease-market shutdown at $\underline{u}$. Numerical simulations of this model suggest that the main difference is that lease prices hit their maximum at the point when all leased minerals are drilled, and then begin to decline as the last few unleased minerals are acquired and drilled. In the current model, lease prices start declining before all leased minerals are drilled. The mass of landowners that are affected substantially by this difference is small.

active leasing, matches arrive at a Poisson rate

$$\theta q(\theta) = \frac{M(\mathcal{U}, \mathcal{V})}{\mathcal{U}}. \tag{8}$$

Since $\sigma q(\theta)$ and $\theta q(\theta)$ are the respective arrival rates of valid matches for firms and landowners when $\mathcal{U} > \underline{u}$, the corresponding mean waiting-times are simply $[\sigma q(\theta)]^{-1}$ and $[\theta q(\theta)]^{-1}$.

One of the properties of the matching function is that each participant creates a positive externality for the opposite side of the market (the thick market effect), and a negative effect on its own side (congestion) (Petrongolo and Pissarides, 2001). Intuitively, if there is lots of unleased acreage available, a firm will be able to quickly find land; however, the presence of many other firms in the market may decrease the rate at which a firm can locate suitable parcels. For example, if many firms are filing leases, this may lead to delays in locating records at the courthouse or processing contracts at the county-level. Alternatively, limited capacity of third-party land-acquisition firms might induce delays. The crucial feature of this model is that the effects of congestion and thick-market externalities are dynamic. Forward-looking landowners and operators will anticipate future market conditions. Firms will lease today to avoid high search costs tomorrow, and landowners gain leverage when tight markets in the future mean it will be easy to sell their land should the current negotiation break down.

Bargaining Let $V^u(t)$ be the selling landowner's value of owning unleased acreage and being in the market. This is a landowners' opportunity cost of leasing. For a firm of type j at time t, its value of operating in the market with a stock of A leases and W drilled wells is $V(A, W, t)$. Denote the marginal value of an additional lease as $\psi_a(t; j) = \frac{\partial V(A,W,Q;j)}{\partial A}$.[14]

The potential gains from trade at time t when a firm of type j matches with a landowner is the difference between the firm's marginal value of a lease $\psi_a(t; j)$ and the landowner's opportunity cost of selling the lease $V^u(t)$. This is the match surplus that the two must choose how (or whether) to split:

$$S(t; j) = \psi_a(t; j) - V^u(t). \tag{9}$$

[14]The firm's value function V will be defined later, and ψ_a will actually be the co-state for the firm's stock of mineral rights. Acemoglu and Hawkins (2006) gives a formal proof of why the marginal shadow-value less lease price is the appropriate value for determining match surplus. In the paper, firms can hire ϵ times a nonnegative integer of workers. The marginal value is obtained as the limit where $\epsilon \to 0$. Stole and Zwiebel (1996a) and Stole and Zwiebel (1996b) derive this bargaining set up as the limit of a multilateral bargaining game in which multiple workers and firms split the surplus match surplus.

I abstract away from the bargaining process by assuming that the price for lease sales satisfies the solution to a generalized Nash bargaining game:

$$p_a(V^u, \psi_a(j)) = \arg\max_{\hat{p}_a} \left[\hat{p}_a - V^u\right]^{\tau} \left[\psi_a(j) - \hat{p}_a\right]^{1-\tau}. \tag{10}$$

The solution to the generalized Nash bargaining problem, p_a, is equal to the landowner's opportunity cost plus her share of the bargaining surplus[15]

$$p_a(t; j) = V^u(t) + \tau S(t; j). \tag{11}$$

In the extreme case where the landowner has all of the bargaining power so that $\tau = 1$, she gets the entirety of the firm's value for the land, $\psi_a(t; j)$. However, because searching is costly and firms would anticipate having to give up their entire value for the lease, no firm would search, and the equilibrium would be degenerate. In the case where the firm has all of the bargaining power, we have that $\tau = 0$, and the landowner gets nothing. I assume that leases can only be procured from landowners by firms, implying that leases are not transferable between firms, and the transactions are irreversible. Thus, a match results in a sale if and only if the surplus is nonnegative: $S(t; j) \geq 0$. The probability that surplus is positive is the probability that a type's valuation for land is greater than the landowner's opportunity cost of leasing: $\Pr\left[\psi_a(t; j) > V^u(t)\right]$.

4.2 Landowners transition functions

Selling landowners are identical, infinitely lived, risk-neutral agents. They own mineral rights but cannot monetize them on their own. Therefore, mineral rights are worthless to the landowner unless there is a possibility that a firm will purchase the rights. When a selling landowner is unmatched, firms arrive at rate $\theta(t)q(\theta(t))$. Should the match surplus with a random firm of type j be nonnegative, the landowner will get payoff $p_a(t; j)$ and give up her value of staying in the market: $V^u(t)$. (The value of being matched is normalized to zero.) If the landowner does not match in a particular period or negotiations do not result in a sale, she will receive flow utility of zero plus her discounted continuation value.

[15] To simplify notation, I drop j and t indexes. The first-order condition for a maximum is

$$\tau(p_a - V^{u\,\tau-1}(\psi_a - p_a)^{1-\tau} - (1-\tau)(p_a - V^u)^{\tau}(\psi_a - p_a)^{-\tau} = 0.$$

Some algebra gives equation (10)

The equation of motion for the landowner's value of being unmatched while $\mathcal{U} > \underline{u}$ is[16]

$$\dot{V}^u(t) = -\theta(t)q(\theta(t)) \int \max\left\{p_a(t;j) - V^u(t), 0\right\} dF(j) + \rho V^u(t).$$

Equation (11) implies that $p_a(t;j) - V^u(t) = \tau S(t;j)$. Define the expected match surplus as

$$\bar{S} \equiv \int \max\{S(j), 0\} dF(j). \tag{12}$$

Then we can write a reduced-form landowner transition equation as

$$\dot{V}^u(t) = \begin{cases} -\tau\theta(t)q(\theta(t))\bar{S} + \rho V^u(t) & \text{if } \mathcal{U} > \underline{u} \\ \rho V^u(t) & \text{otherwise.} \end{cases} \tag{13}$$

The equation illustrates that each period, the landowner has some probability of being matched, $\theta(t)q(\theta(t))$, and receiving her share of the surplus, $\tau S(t;j)$. The surplus term enters with a negative sign because the passage of one period eliminates one opportunity for the landowner sell her land.

Define the time when firms stop searching forever as $T_0 = \{\min t | v(t;j) = 0 \forall t > T_0, \forall j\}$. Since an unmatched landowner will never sell her land after T_0, the opportunity cost of leasing at $t = T_0$ must be zero. This provides the following boundary condition for V^u:[17]

$$V^u(T_0) = 0. \tag{14}$$

Since we have bounded unleased acreage away from zero, $\mathcal{U} > 0$, market tightness and the arrival rate of matches, $\theta q(\theta)$, will be finite. Aggregate surplus $\bar{S}$, will also be finite if firms' lease valuations, $\psi_a(t;j)$, and landowners' opportunity costs, $V^u(t)$ are finite. This implies that $\dot{V}^u(t)$ is well-defined. Therefore, $V^u(t)$ exists and is continuous.

Preliminary analysis of landowners There are several things to note about landowners' opportunity cost of leasing. First, the solution to the Nash bargaining problem, equation (10), means that it is always optimal for a landowner to sell since lease prices are always at least as big as the opportunity cost of leasing: $p_a(t;j) \geq V^u(t) \forall t, j$.

Second, the transversality condition that the last, unleased landowner has zero oppor-

[16] This is derived using dynamic programming in Appendix B.1.

[17] Note that any notion of a steady state requires that $V^u(T_0) = 0$ because equation (13) becomes $\dot{V}^u(t) = \rho V^u(t)$ when $v(t;j) = 0 \forall j, t > T_0$ and $\dot{V}^u(t) = 0$ if and only if $V^u(t) = 0$.

tunity cost of leasing, equation (14), means that landowners' opportunity cost is falling just before leasing stops: $\exists \epsilon > 0$ such that $\dot{V}^u(t) < 0$ for $t \in (T_0 - \epsilon, T_0]$. The intuition is that when leasing is about to stop, every period when a landowner cannot lease reduces her likelihood of monetizing her land in the future and, hence, lowers her opportunity cost of leasing.

Third, landowners' opportunity cost of leasing is always positive for $t < T_0$ as long as firms have non-zero valuations for the land. To see this, integrate equation (13) and use the transversality condition, equation (14), to compute landowners' opportunity cost as

$$V^u(t) = \begin{cases} e^{\rho t} \int_t^{T_0} e^{-\rho s} \left[\tau \theta q(\theta) \bar{S} \right] ds & \text{if } t \leq T_0 \\ 0 & \text{if } t \geq T_0 \end{cases}. \tag{15}$$

The price that the landowner receives from a firm of type j, $p_a(t;j) = V^u(t) + \tau S(t;j)$, explicitly compensates her for the opportunity cost of leasing. However, this compensation has nothing to do with disutility drilling imposes on the landowner. One could view landowners' transition equation, equation (13), as an asset-pricing equation. Then the first term in equation (13) takes the place of a dividend, and the second term represents the discounted value of the asset.

Fourth, since the value of landowners' opportunity cost, V^u, rises slower than the rate of interest, its present value is always falling. The fact that the present-value is falling can be easily shown by discounting equation (15) and differentiating with respect to time:

$$\frac{d}{dt} \left[e^{-\rho t} V^u(t) \right] = -\tau \theta q(\theta) \bar{S} < 0.$$

If we view the landowners' acreage as an asset, this implies that it is optimal for the landowner to sell as soon as she can. *This will hold even when there are future technological improvements* because these will be accounted for in firms' valuations of the leases, $\psi_a(t;j)$, as we will see in Section 5.2.2 on page 36. Thus, any *ex-post* regret landowners have over selling early for a low price can only be explained by *unexpected* developments in price or technology.

Fifth, the current-value of leases are not always falling. In fact, equation (13) shows that $\dot{V}^u$ can be positive if landowners' match arrival rates are low because the leasing market is loose or the surplus is small. Both will diminish the negative term in the equation (13) for $\dot{V}^u$ and signal that markets may be tighter and the surplus, larger in the future.

4.3 Operators

Forward-looking, competitive, and profit-maximizing operators are in the business of searching for and purchasing leases, drilling wells on them, and selling the resources that flow from the wells. Each operator is characterized by a type j, distributed as $j \sim F(j)$, a stock of acres it has leased in perpetuity, $A(t;j)$; a stock of wells it has drilled, $W(t;j)$; and the current flow of production $Q(t;j)$ from the wells it has drilled.

Search for leasing Operators choose a path for the intensity of their search for acreage, $v(t;j)$, which they purchase at a negotiated price, $p_a(t;j)$ (discussed earlier). In choosing their search intensity, operators take as given the paths of aggregate searching, $\mathcal{V}(t)$; the quantity of unleased acreage, $\mathcal{U}(t)$; and landowners' opportunity cost of selling, $V^u(t)$. Each firm matches with selling landowners at a rate proportional to its search effort. Thus, an individual firm's flow of matches with selling landowners, which is also the quantity of acres it purchases, is

$$\dot{A}(t;j) = \begin{cases} v(t;j)\sigma(t)q(\theta(t)) & \text{if } \psi_a(t;j) \geq V^u(t) \text{ and } \mathcal{U} > \underline{u} \\ 0 & \text{otherwise.} \end{cases} \tag{16}$$

Searching must be nonnegative because leasing is irreversible, and firms cannot sell leases to other firms. This imposes the constraint $v(t;j) \geq 0$.

Drilling wells Operators also choose a path for the rate at which they drill wells, $w(t;j)$, on their stock of accumulated leases, $W(t;j)$. The stock of drilled wells increases at the drilling rate: $\dot{W}(t;j) = w(t;j)$, and drilling is irreversible: $w(t;j) \geq 0$. Additionally, the firm can only drill wells on land that it has leased, and each lease can hold a single well. Thus, cumulative leasing must always be at least as large as cumulative drilling: e.g., $A(t;j) - H(t;j) \geq 0 \forall j, t$.[18] The fact that leases are homogeneous is important for tractability since it means there is only one lease-before-drill constraint. Heterogeneity of leases in quantity of recoverable resource, cost, or lease expiration date would imply a separate lease-before-drill constraint for each type of lease. Production from a well starts at

[18] One could imagine that leases are sold in "well-sized" tracts, so A is the acreage sufficient for A wells. This easily generalizes to multiple wells. For example, firms sometimes re-fracture wells or drill a second well to a different geological layer, which would appear to contradict the restriction that $A(t;j) - H(t;j) \geq 0$. To incorporate this phenomenon, one can simply rescale A.

a possibly type-dependent initial production rate $q_0(j)$ and declines exponentially at rate δ_q. An operator sells its flow of production, $Q(t;j)$, in the market at a constant, exogenous price p_q.[19]

To drill wells at rate w, firms must purchase (demand) a quantity of drilling services $c^d \geq c(w, A, W, \mathcal{W}; j)$ from a competitive drilling services industry at price $p_c(t)$, which operators take as given. The cost function (in terms of drilling services) depends on the firm's type, j; drilling rate, w; stock of acres, A; cumulative drilling, W; and cumulative drilling by the industry as a whole $\mathcal{W}$. I include cumulative leasing, cumulative drilling, and cumulative aggregate drilling to capture the possibilities of rising costs due to depletion, learning-by-doing, and knowledge spillovers. In particular, making cost dependent on acreage, A, captures the negative effect on costs that owning more acreage could have on drilling. This could be a scale effect in which larger operators can take advantage of fixed infrastructure. It could also be part of an economic depletion effect in which resource extraction leads to progressively lower-quality deposits. In that case, owning more acreage would give access to untouched and low-cost deposits. Assumption 5.3 describes the basic assumptions on $c(\cdot)$.

Optimization problem An operator of type j with initial stock of acres $A_0(j)$, stock of drilled wells $W_0(j)$, and production $Q_0(j)$, given prices p_q and $p_c(t)$ as well as landowners' opportunity costs $V^u(t)$ must to choose paths $v(t;j)$ and $w(t;j)$ to maximize the present value of its profits. This is the following dynamic program. (To simplify notation, I omit the explicit dependence of prices, states, and controls on time, t, and I include type, j, only when the dynamic program's primitive parameters are type-dependent.)

$$\widetilde{V}(A_0, H_0, Q_0; j) = \max_{w,v} \int_0^\infty e^{-\rho t} \left\{ p_q Q - p_c c(w, A, W, \mathcal{W}; j) - \kappa(v) - p_a(V^u, V^m(j), \psi_a)\dot{A} \right\} dt \quad (17)$$

[19] This setup departs from reality in two simple ways. First, it implies that operators cannot control the rate of production from a well once it is drilled. Though operators are able generally able to choke back shale production, as Anderson, Kellogg, and Salant (2014) shows, it is not optimal to do so in almost all cases. Second, though shale wells do exhibit production declines, the declines are not exponential, as shown by Patzek, Male, and Marder (2013). The exponential decline implies a convenient analytical form for the present value of revenues from drilling a well and could be readily modified.

subject to the constraints

$$\dot{A} = v\sigma q(\theta)\mathbb{1}\left[\mathcal{U} > \underline{u}\right] \tag{16}$$

$$\dot{W} = w \tag{18}$$

$$\dot{Q} = q_0(j)w - \delta_q Q \tag{19}$$

$$A - W \geq 0 \tag{20}$$

$$v \geq 0 \tag{21}$$

$$w \geq 0 \tag{22}$$

$$(A(0;j), W(0;j), Q(0;j)) = (A_0(j), W_0(j), Q_0(j)) \tag{23}$$

$$\psi_a \equiv \frac{\partial V(A,H,Q;j)}{\partial A}. \tag{24}$$

While equation (17) is an infinite horizon problem, depletability will mean that we can modify the it so that firms choose a depletion time T_j, after which they simply receive the discounted present value of their remaining production.

Operator entry The mass of operators active in a shale is determined by a free entry condition. Before entry, operators have no leases, wells, or existing production. They pay a common fixed cost C_{entry} to enter the market, and then they learn their type j. At time $t = 0$, the mass of firms in the market, n, is such that the expected profit of entry is nonpositive:

$$C_{\text{entry}} \geq \int \widetilde{V}(0,0,0;j)dF(j). \tag{25}$$

4.4 Services

At time t, a unit measure[20] of identical, competitive service firms each supply $c^s(t)$ units of drilling services into a competitive market at a market price $p_c(t)$.[21] They each have a stock of long-lived capital $K(t)$, which they can purchase or sell at rate $i(t)$. The capital stock depreciates at rate δ_k. The services company combines labor, $L(t)$, and capital, $K(t)$, to produce drilling services using a production technology $f(L,K)$. Labor is purchased in a perfectly elastic labor market with constant wage p_l, and capital is purchased in

[20] As with operators, it is straightforward determine the mass of service firms using a free-entry condition.

[21] Note, to simplify the problem, one can easily eliminate the services sector and set $p_c(t) = \overline{p}_c \forall t$. This eliminates some interesting dynamics from the problem but reduces the number of differential equations that must be solved.

competitive market at price $p_k(t)$. After all drilling stops at an equilibrium-determined time $t = T = \max\{T_j\}_{j \in J}$, the remaining capital stock, $K(T)$, is sold in the market at price $p_k(T)$ (its scrap value).

Given an initial stock of capital and prices for labor, capital, and services, these firms solve the following dynamic program. (As before I omit explicit dependence on time to simplify notation.)

$$V^s(K) = \max_{L,i} \int_0^T e^{-\rho t} \{p_c f(L, K) - p_l L - p_k i\} \, dt + e^{-\rho T} p_k(T) K(T) \tag{26}$$

subject to

$$\dot{K} = i - \delta_k K \tag{27}$$

$$K(0) = K_0. \tag{28}$$

4.5 Market clearing

In addition to the definition of aggregate searching, any equilibrium must satisfy two market clearing conditions at all times. First, services demand must equal supply:

$$n \int c^d(t; j) dF(j) = c^s(t). \tag{29}$$

Second, the demand for new capital, $i(t)$, by oilfield services firms must equal the supply of new capital $\mathcal{I}(t)$. The price of capital is determined from an increasing inverse supply function::

$$p_k(t) = p_k(\mathcal{I}(t)). \tag{30}$$

5 Equilibrium

In this section, I define the notion of a dynamic equilibrium. Then I characterize the ordinary differential equations (ODEs) that govern its evolution for the case when all leasing stops before any firm ceases to drill. Given a set of stopping times and ending values for states and co-states, it is possible to show existence and uniqueness of the equilibrium path as these are characterized by well-behaved ODEs. However,

5.1 Equilibrium definition

An equilibrium is a tuple

$$\Big\langle A_0(j), W_0(j), Q_0(j), K_0, F(j), V^u(T_0), i(t), c^s(t), v(t;j), w(t;j), \\ V^u(t), p_k(t), p_c(t), \mathcal{V}(t), \mathcal{U}(t), p_a(t;j), n\Big\rangle$$

which is defined for all times $t \in [0, \infty]$ and types $j \in J$. The tuple includes the following elements: exogenously given initial stocks for operators and oilfield services, the distribution of firms, and a terminal condition for landowners' opportunity cost of leasing: $\{A_0(j), W_0(j), Q_0(j), K_0, F(j), V^u(T_0)\}$; investment and production decisions by oilfield services, $\{i(t), c^s(t)\}$; search intensity and drilling decisions for each type, $\{v(t;j), w(t;j)\}$; landowners' opportunity cost of leasing, $\{V^u(t)\}$; market prices for capital and services, $\{p_k(t), p_c(t)\}$; lease market tightness and unleased acreage, $\{\mathcal{V}(t), \mathcal{U}(t)\}$; lease prices paid by each type of firm, $\{p_a(j)\}$; and the mass of operators, n such that the following criteria are satisfied. First, firms' controls (investment, production, searching, drilling) solve (17) and (26) subject to their respective constraints (*optimality*); second, *landowners' opportunity cost* of leasing satisfies transition equation (13); third, capital and services prices clear their respective markets (*market clearing* equations (30) and (29)); fourth, aggregate searching satisfies (4); and lease-market tightness, (6) (*lease market*); fifth, lease prices, p_a, satisfy *generalized Nash bargaining* (10); and sixth, *free entry* occurs at $t = 0$ until operators' profits are nonpositive (25). Several variables have been omitted from the equilibrium definition since they are implied by the equilibrium path. First, stocks are implied by the controls plus initial conditions. Second, market tightness, $\theta(t)$, is implied by $\mathcal{U}(t)$ and $\mathcal{V}(t)$.

5.2 Characterization of the equilibrium

In this section, characterize the equilibrium path of a shale boom. First, I lay out several assumptions, mostly on the smoothness of various functions. Under these assumptions, I argue that unique optimal controls exist for operators and oilfield services firms given prices, states, and co-states. This implies that landowners' value functions are well-defined and continuous. Second, I argue that for every possible combination of states and landowner opportunity costs, there exist market-clearing prices in the services and capital markets. Third, I use the definition of aggregate searching to pin down firms' equilibrium search effort and leasing rates. We are left with a boundary value problem (BVP) composed of a set of ordinary differential equations (ODEs) and transversality conditions associated with starting and ending times. The ODEs are well behaved and characterize a unique path given a set of either initial or terminal state and co-state values. To show existence and uniqueness of the equilibrium path, all that remains corresponding to a particular set of starting values for the stocks of mineral leases, drilled wells, and capital. Unfortunately, I am not able to analytically show existence and uniqueness of these values, which would, together with the ODEs, constitute a unique solution to the BVP and, therefore, the equilibrium path.[22] I am, however, able to obtain what appear to be unique numerical solutions to the model, suggesting that a unique equilibrium path exists.

Assumption 5.1. *For all $j \in J$, the leasing constraint $A(t;j) - W(t;j) \geq 0$ only binds after all leasing activity has stopped at $t = T_0$ where $T_0 = \{\min t | \mathcal{U}(t) = \underline{u}\}$.*

Assumption 5.1 restricts attention to equilibria in which aggregate leasing stops before firms are able to drill all of their acreage. Though this is somewhat artificial, it considerably simplifies analysis and allows for an analytic characterization of the solution because it implies the drilling and leasing problems do not interact until the a firm drills its very last well. While this assumption can be relaxed, doing so comes at a loss of analytical tractability.

I now make several smoothness and monotonicity assumptions on cost and production functions.

Assumption 5.2. *The cost function $\kappa(\cdot)$ is strictly increasing, convex, and twice-differentiable. Additionally, $\lim_{v\downarrow 0} \kappa'(v) = 0$ and $\lim_{v\uparrow\infty} \kappa'(v) = \infty$.*

Assumption 5.3. *$c(w, A, W, \mathcal{W}; j) \in \mathbb{C}^2$ is well-defined and strictly convex over the the state space*

$$\{(w, A, W, \mathcal{W}; j) | w \geq 0, A > 0, W \in [0, A), \mathcal{W} \leq \mathcal{A}, j \in J, \},$$

and, if $c(\cdot; j)$ is not defined where $A = W$, then $\lim_{A\to W} c(w, A, W, \mathcal{W}; j) > \frac{p_q q_0(j)}{\rho + \psi_q}$. Drilling costs are strictly increasing and convex in the drilling rate: $c_{ww}, c_w > 0$. The cost of drilling nothing is zero: $c(0, A, W, \mathcal{W}; j) = 0$, and infinite drilling imposes infinite cost:

[22] The competitive problem is not equivalent to the social planner's problem due to search externalities, so the usual approach of obtaining existence and uniqueness from the social planner's problem is not applicable here. It is possible to make the ODEs monotonic in the states by removing the non-selling landowners (setting $\mathcal{A}^n = 0$), which implies uniqueness of the solution to the BVP. Unfortunately, this does not guarantee existence. In another paper, I am pursuing this modification and removing the lease-market shutdown at $\underline{u}$.

$\lim_{w\to\infty} c(w, A, W, \mathcal{W}; j) = \infty$. *Adding acreage cannot increase costs:* $c_A \leq 0$. *Finally, if at any point drilling increases costs,* $c_W > 0$, *then leasing must decrease them:* $c_A < 0$.

The stock of mineral leases is included in order to capture the possibility of economic depletion. If the cost function exhibits economic depletion, we will need costs to rise with the percentage of the resource base (captured by the stock of mineral leases) that has been drilled. A firm can increase its resource base by purchasing mineral rights, and it decreases them by drilling more. Thus, anytime we have costs rising with drilling, $c_W > 0$, we must also be able to decrease them by purchasing unexploited acres: $c_A \leq 0$.

Assumption 5.4. *The inverse supply curve of capital is strictly increasing:* $p_k' > 0$. *Also* $\lim_{\mathcal{I}\to-\infty} p_k(\mathcal{I}) = 0$ *and* $\lim_{\mathcal{I}\to\infty} p_k(\mathcal{I}) = \infty$. *(Note this also implies that* $p_k > 0 \forall \mathcal{I} > -\infty$.*)*

Assumption 5.5. *The production function is such that* $f_L, f_K, f_{LK} > 0$ *and* $f_{LL}, f_{KK} < 0$. *Also,* $f(\cdot,\cdot)$ *exhibits constant returns to scale and* $f(0, K) = f(L, 0) = 0$. *Furthermore,* $\lim_{L\downarrow 0} f_L = \lim_{K\downarrow 0} f_K = \infty$, *and* $\lim_{L\uparrow\infty} f_L = \lim_{K\uparrow\infty} f_K = 0$

Assumption 5.6. *The resource price, depletion rate, initial production rate, and discount rate are such that zeroth marginal well is profitable for all firm types* j: $\frac{p_q q_0(j)}{\rho+\delta_q} - c_w(0, A, 0, \mathcal{W}; j) > 0 \forall A > 0, \mathcal{W} \geq 0, j \in J$

Though not necessary, this assumption removes the possibility of a degenerate equilibrium in which firms do not drill.

Assumption 5.7. *The sets of admissible states for operators,* $\{(A, H, Q)\}$, *and oilfield services firms,* $\{(K)\}$ *are bounded. Additionally, given any admissible state, the sets of all admissible controls for operators,* $\{(v, w)\}$, *and oilfield services,* $\{i\}$, *are bounded.*

Assumption 5.8. *There are a finite number of discrete types:* $|J| < \infty$.

This assumption guarantees that the equilibrium is characterized by a finite number of equations.

5.2.1 Existence and uniqueness of optimal controls

We can use optimal control methods to characterize the equilibrium path. Since the operators' problem involves pure state-constraints, we must turn to slightly more general statements of the Maximum Principle than are generally needed for problems with constraints on the controls only (Hartl, Sethi, and Vickson, 1995). Assumption 5.7 together with the fact that operators' and oilfield services firms' profit functions are bounded implies that the assumptions of Theorem 3.1 in Hartl, Sethi, and Vickson (1995, p. 185) are satisfied. Therefore, given prices and market conditions, paths for optimal controls and states exist for problems (17) and (26). Theorem 4.1 in Hartl, Sethi, and Vickson (1995, pp. 186-187) provides necessary conditions for the optimal path that must be satisfied by any equilibrium. Assumptions 5.2 and 5.3 mean that operators' costs are strictly convex in the states and controls and that state transition functions are linear functions of the controls. Assumption 5.5 implies that the service firm's production function is strictly concave. Thus, the Hamiltonians for both operators and service firms are strictly concave in both states and controls. Then we can apply Theorem 8.3 and Corollary 8.2 of Hartl, Sethi, and Vickson (1995, p. 203) to show that the necessary conditions are sufficient and characterize unique optimal paths.

Though state-constraints can cause discontinuities in the co-states, for the operators' problem, they do not as long as prices and market tightness are continuous functions of time. Later in this section, we will demonstrate that they must be (except for $\mathcal{V}$ when leasing stops at T_0). Strict convexity of the Hamiltonian implies that the maximizers v^* and w^* are unique. Then, if prices and aggregate variables are continuous, Proposition 4.3 of Hartl, Sethi, and Vickson (1995, p. 191) implies that the optimal controls are continuous functions of time as long as prices are, too. Since the first time derivative of the state constraint $A - W \geq 0$ depends directly on controls, the assumptions of Proposition 4.2 are met. So, co-states are also continuous at the junction time(s) when the constraint just starts to bind or stops binding.

5.2.2 Operator optimality

Appendix B.2 shows how firms form a finite-horizon optimization problem equivalent to the infinite horizon problem (17). In the finite-horizon formulation, the production state variable, Q, is removed. Now, instead of explicitly accounting for production revenues from all past wells in each time period, firms simply look at the net present value of revenues from drilling.[23] Define the net present value of expected revenues from drilling as

$$\widehat{p}(j) \equiv \frac{p_q q_0(j)}{\rho + \delta_q}. \tag{32}$$

Given initial conditions A_{0j} and W_{0j}, the price of services, $p_c(t)$, and conditions in the lease market, $\{\mathcal{V}(t), \mathcal{U}(t), V^u(t)\}$, the firm's problem is to choose paths for drilling, $w(t;j)$; searching, $v(t;j)$; the time when drilling stops, $T_j = \{\min t | w(t;j) = 0 \forall t > T_j\}$ (possibly infinite); a terminal lease portfolio, A_{T_j}; and terminal cumulative drilling W_{T_j} to maximize the sum of revenues from existing production plus profits from new production. This is

$$\widetilde{V}(A_{0j}, W_{0j}, Q_{0j}; j) = \frac{p_q Q_{0j}}{\rho + \delta_q} + V(A_{0j}, W_{0j}; j)$$

where the we can write firms' optimization problem for its new production as

$$V(A_{0j}, W_{0j}; j) = \max_{w(t), v(t), T_j, A_{T_j}, W_{T_j}} \int_0^{T_j} e^{-\rho t} \left\{ \widehat{p}(j) w - p_c c(w, A, W, \mathcal{W}; j) - \kappa(v) - p_a(V^u, V^m(j), \psi_a) \dot{A} \right\} dt \tag{33}$$

subject to constraints (20), (16), (22), (21), and (24).

Necessary and sufficient conditions To solve the problem, we can use optimal control methods. After substituting in equation (16) for the flow of leases, $\dot{A}$, the current-value Hamiltonian corresponding to equation (33) is

$$\mathcal{H} = \hat{p}(j) w - p_c c(w, A, W, \mathcal{W}; j) - \kappa(v) + (\psi_a - p_a) v \sigma q(\theta) + \psi_w w. \tag{34}$$

The current-value Lagrangian is

$$\mathcal{L} = \mathcal{H} + \lambda(A - W) + \lambda_w w + \lambda_v v.$$

[23] The path of optimal production, $Q(t;j)$, can be easily computed by integrating the law of motion for for production, equation (19), and using the initial condition $Q(0;j) = Q_{0j}$:

$$Q(t;j) = e^{-\delta_q t} \int_0^t e^{\delta_q \tau} q_0 w(\tau; j) d\tau + Q_{0j} e^{-\delta_q t}. \tag{31}$$

There are several necessary (and, as argued previously) sufficient conditions for an optimum (Hartl, Sethi, and Vickson, 1995). The first-order conditions for the optimal paths of searching and drilling are[24]

$$\frac{\partial \mathcal{L}}{\partial w} = -p_c c_w(w, A, W, \mathcal{W}; j) + \psi_w + \widehat{p}(j) + \lambda_w = 0$$
$$\frac{\partial \mathcal{L}}{\partial v} = -\kappa'(v) + (\psi_a - p_a)\sigma q(\theta) + \lambda_v = 0.$$

The co-state equations are

$$\dot{\psi}_w = p_c c_W(w, A, W, \mathcal{W}; j) + \lambda + \rho\psi_w \tag{35}$$
$$\dot{\psi}_a = p_c c_A(w, A, W, \mathcal{W}; j) - \lambda + \rho\psi_a. \tag{36}$$

The complementarity conditions for the constraints are

$w \geq 0$	$\lambda_w \geq 0$	$w\lambda_w = 0$	Drill irreversibility
$v \geq 0$	$\lambda_v \geq 0$	$v\lambda_v = 0$	Lease irreversibility
$A - W \geq 0$	$\lambda \geq 0$	$\lambda(A - W) = 0.$	Lease before drill

The optimality condition for T_j is that the maximized current-value Hamiltonian has a value of zero at T_j:

$$\mathcal{H}^*(A(T), W(T), T_j; j) = 0. \tag{37}$$

Initial stocks of leases and drilling are both fixed at zero: $A_0(j) = W_0(j) = 0$, but terminal values, $A_T(j)$ and $W_T(j)$, are free variables. Neither the acreage portfolio, A, nor cumulative drilling, W, has any scrap value. Theorem 4.1 of Hartl, Sethi, and Vickson (1995, pp. 186–187), implies that we must have a multiplier ν for the state-constraint $A - W \geq 0$ such that

$$\psi_a(T_j; j) = \nu \tag{38}$$
$$\psi_w(T_j; j) = -\nu \tag{39}$$
$$\nu \geq 0 \tag{40}$$
$$\nu(A - W) \geq 0 \tag{41}$$

[24] With the cost-function $c(w, A, W, \mathcal{W}, ; j)$, I use sub-scripts to indicate partial derivatives. This is also done later in the paper when the cost-function is decomposed into constituent components.

As noted previously, Proposition 4.2 of Hartl, Sethi, and Vickson (1995, pp. 190–191) implies that the co-states, ψ_a and ψ_w, are continuous if and when the constraint $A - W \geq 0$ binds.[25] Thus, we do not need to consider jumps in the co-states.

Interpreting operators' FOCs The first FOC implies that the present value of revenues from a well must equal the marginal extraction cost (MEC) plus the opportunity cost of drilling, also known as the marginal user cost (MUC) (Medlock, 2009):

$$\widehat{p}(j) \leq \underbrace{p_c c_w(w, A, W, \mathcal{W}; j)}_{\text{MEC}} + \underbrace{(-\psi_w)}_{\text{MUC}}. \tag{42}$$

The opportunity cost of drilling a well is simply the present value of the profits that could be obtained were the operator to delay drilling the last well and instead drill it in the subsequent period. The second FOC shows that the marginal cost of leasing must equal the producers' share of match surplus:

$$\kappa'(v) = (1 - \tau) \max \{S, 0\} \, \sigma q(\theta). \tag{43}$$

Interpreting operators' co-state equations Operators' co-states, ψ_a and ψ_w, capture the value of an additional lease and the opportunity cost of drilling, respectively. Assumption 5.1 implies $\lambda(t) = 0 \forall t < T_j$ and allows us to integrate equations (36) and (35) to obtain

$$\psi_a(t; j) = e^{-\rho(T_j - t)} \psi_{aT}(j) - e^{\rho t} \int_t^{T_j} e^{-\rho s} p_c(s) c_A(s) ds$$
$$\psi_w(t; j) = e^{-\rho(T_j - t)} \psi_{wT}(j) - e^{\rho t} \int_t^{T_j} e^{-\rho s} p_c(s) c_W(s) ds.$$

This shows that the co-states are equal to the present value of their terminal values less the present value of the marginal change in future drilling costs that adding an additional lease or well will cause. For example, if learning-by-doing is present, then drilling an additional well yields marginal benefits beyond just the well's accounting profits. Learning lowers future costs and makes $c_W < 0$. Thus, learning effects will tend to increase ψ_w. In some

[25] Strictly speaking, Proposition 4.2 requires that a rank condition is satisfied. This technically fails since we constrain w from above ($A - W \geq 0 \implies \dot{A} - \dot{W} \geq 0 \implies w \leq 0$), as well as below $w \geq 0$. However, we can safely drop the non-negativity constraint on $w(t; j)$ since the state-constraint, not the control constraint, binds at the very end, and "undrilling" wells would b

ways, learning-by-doing acts exactly like capital accumulation, which, as we will see, lowers short-run marginal costs of production. A significant difference between the two is that learning does not depreciate, while capital does.

Transversality conditions To fully determine firms' optimal paths given prices and market conditions, we require transversality conditions for $A_T(j)$, $W_T(j)$, T_j, $\psi_{aT}(j)$, and $\psi_{wT}(j)$. First, note that the transversality conditions for $A_T(j)$ and $W_T(j)$, equations (38) and (39), together imply that $\psi_{aT}(j) = -\psi_{wT}(j)$. This means that when costs do not change with the quantity of acreage purchased, ($c_A = 0$), the only link between the value of drilling and the value of leasing happens when the firm exhausts its leased minerals at $t = T_j$. The disconnect between firms' shadow-values for leases and wells is important because it means that all foreseen technological progress is priced into leases from the very beginning at $t = 0$. Thus, even if learning-by-doing impacts the opportunity cost of drilling, $\psi_w(t;j)$, along its entire path, learning only affects the value of leases through its effect on $\psi_{wT}(j)$. In this case, we can write $\psi_a(t;j) = -e^{-\rho(T_j-t)}\psi_{wT}(j)$. This equation makes it clear that lease valuations are changed both by the time at which the constraint binds, T_j, and the terminal rent, $-\psi_{wT}(j)$. Learning-by-doing that lowers costs can only affect lease-valuations through either hastening the time of depletion, T_j, and, the discount applied to the terminal rent, or the increasing opportunity cost of drilling the last well, $\psi_{wT}(j)$. Thus, landowners who lease at $t = 0$ are still the beneficiaries of future technological change that increases profitability. Learning will also affect lease values in the case of economic depletion by hastening depletion and changing the cost function; however, these effects are more muddied and less readily apparent.

We will now attack the optimal time of depletion, T_j, which is determined by equation (37) and then classify the problem into two types to finish characterizing the transversality conditions. Recall that Assumption 5.1 implies that leasing stops before any firm drills all of its mineral leases: $T_0 < T_j \forall j$. This implies that $\dot{A}(T_j;j) = v(T_j;j) = 0$. Equation (37) then implies that we can set

$$\mathcal{H} = p(j)w - p_c c(w, A, W, \mathcal{W}; j) + \psi_w w = 0.$$

Suppose that $w(T_j) > 0$. Substituting in the FOC for drilling, equation 42 implies that

$$p(j)w - p_c c(w, A, W, \mathcal{W}; j) - \left[\widehat{p}(j) - p_c c_w(w, A, W, \mathcal{W}; j)\right] w = 0.$$

In turn, this implies that

$$c(w, A, W, \mathcal{W}; j) = c_w(w, A, W, \mathcal{W}; j)w.$$

Assume for the sake of contradiction that $w(T_j) > 0$. Then marginal drilling cost equals average drilling cost (simply divide both sides by w). Assumption 5.3 that $c(\cdot)$ is strictly convex implies the statement can only be true when $w = 0$, which is a contradiction. Therefore it must be that the optimal drilling rate at $t = T_j$ is $w(T_j) = 0$. It is not optimal for the firm raise the rate of extraction and to accelerate production so that drilling drops discontinuously from a positive value to zero at time T_j. Strict convexity of the cost function implies that such an accelerated depletion schedule would increase costs and lower profit compared to the case where T_j is free. Having determined $w(T_j)$, we can use the FOC for drilling, equation (42), to see that $\psi_w(T_j; j) = \psi_{wT}(j)$ must be the present value of well production less the marginal cost of drilling the last well:

$$-\psi_{w,T_j} = \widehat{p}(j) - p_c(T_j)c_w\left(0, A_{T_j}, W_{T_j}, \mathcal{W}(T_j); j\right). \tag{44}$$

We still have yet to determine terminal leasing and drilling $A_T(j)$ and $W_T(j)$. There are two cases to consider: economic depletion and physical depletion. A resource is said to be *physically exhausted* when it is optimal for the firm to exploit all of it: $A_T(j)^* = W_T(j)^*$, and it is is said to be *economically exhausted* when it is optimal for the firm leave some leases undrilled: $A_T(j)^* \geq W_T(j)^*$. Physical depletion implies that the multiplier on the lease-before-drill constraint, equation (20), binds: $\lambda(T_j) > 0$. Economic depletion implies that it does not, so $\lambda(T_j) = 0$.

The complementarity constraint for the transversality conditions, equation (41), implies that when we have economic depletion, $\lambda(T_j) = \nu(T_j) = 0$, which means

$$\psi_{a,T_j} = -\psi_{w,T_j} = 0.$$

In other words, economic depletion is characterized by a situation in which the last well drilled makes zero profit. Because of this and Assumption 5.3 about the continuity of the drilling cost function, $c()$, there is no opportunity cost for drilling the last well, and no marginal benefit to adding enough acreage to drill another zero-profit well. Thus, acreage and cumulative drilling have scrap values of zero. Given $A_T(j)$ and the convexity assumptions on $c(\cdot)$, we can determine $W_T(j)$ from equation (44):

$$0 = \widehat{p}(j) - p_c(T_j)c_w\left(0, A_T(j), W_T(j), \mathcal{W}(T_j); j\right).$$

With physical exhaustion, we know that the multiplier on the transversality conditions is positive, and $\lambda > 0 \implies \nu > 0$. We then use the fact that $A_T(j) = W_T(j)$ and equation (44) drilling to determine the values of $\nu(T_j)$, $\psi_{aT}(j)$, and $\psi_{wT}(j)$:

$$\nu(T_j) = \psi_{aT}(j) = -\psi_{wT}(j) = \widehat{p}(j) - p_c(T_j)c_w\left(0, A_T(j), A_T(j), \mathcal{W}(T_j); j\right) > 0.$$

Finally, A_{T_j} is determined not by the firm in isolation, but by the history of its previous searching behavior and the history of lease market conditions. The driver of the firm's searching is ψ_a, which is determined by the transversality conditions and its equation of motion, equation (36).

5.2.3 Service firms' optimal behavior

The optimization problem for oilfield services firms is simpler than operators' because it has only one state variable and no state-constraints. It will be convenient to work with a labor-demand function instead of a production function. Assumption 5.5 implies that we can implicitly define the labor-demand function, $L(c^s, K)$, as

$$f(L(c^s, K), K) = c^s.$$

At the terminal time for oilfield services, the firm stops providing drilling services and sells off its remaining capital stock. The oilfield services terminal time is determined by the operator type which finishes last: $T = \max_{j\in J}\{T_j\}$. Oilfield services firm take this time as given, so it does not require a transversality condition. However, the terminal capital stock is not exogenously set, just the starting capital stock. So, to complete the description of the oilfield services optimal control problem, we require an transversality condition that incorporates the scrap-value of capital. This is specified below.

After substituting in the implicit labor-demand function, the current value Hamiltonian corresponding to the representative service firms' problem, equation (26), is

$$\mathcal{H}^s = p_c c^s - p_l L(c^s, K) - p_k i + \psi_k(i - \delta_k K).$$

The FOC are

$$\frac{\partial \mathcal{H}^s}{\partial c^s} = p_c - p_l L_c = 0$$
$$\frac{\partial \mathcal{H}^s}{\partial i} = -p_k + \psi_k = 0.$$

The first FOC implies that the short-run marginal cost of providing drilling services (labor costs) equals the marginal revenue:

$$p_c = p_l L_c(c^s, K). \tag{45}$$

The second FOC implies that the sector purchases capital until the marginal cost of capital equals its shadow value:

$$p_k = \psi_k. \tag{46}$$

The co-state equation shows that the shadow-value of capital is derived from the cumulative reductions in short-run marginal cost (labor costs) capital provides, less depreciation and time-discounting:

$$\dot{\psi}_k = p_l L_k + (\rho + \delta_k)\psi_k. \tag{47}$$

If we integrate this equation, captial's becomes even clearer:

$$\psi_k(t) = e^{-(\rho+\delta_k)(T-t)}\psi_{kT} + e^{(\rho+\delta_k)t}\int_t^T e^{-(\rho+\delta_k)s} - p_l L_k ds.$$

The price and quantity of drilling services is determined in equilibrium, as is the transversality condition for the terminal capital stock. This is a free variable, K_T, and is all sold off in the market. The equilibrium condition will be the same as equation 46:

$$p_k(T) = \psi_{kT}.$$

Market clearing in the capital market will imply that at the last time of active drilling we have

$$p_k(T) = p_k(-K_T).$$

This completes characterization of the optimal path for oilfield services.

5.2.4 Markets clear

Lemma 5.1. *Given any admissible set of states and co-states at time t, we can find a price $p_c(t)$ to clear the drilling services market, and $p_c(t)$ is a continuous function of t.*

The proof, given in Section B.3 is constructive and makes heavy use of Assumptions 5.3 and 5.5 about the smoothness and convexity of drilling costs and the smoothness and concavity of the drilling services production function.

Lemma 5.2. *Given any demand for capital $i \in \mathbb{R}$, there exists a unique market-clearing price of capital, p_k, and $p_k(t)$ is continuous.*

Proof. This follows immediately from Assumption 5.4 that the marginal supply function is strictly increasing and is defined for all the real numbers. The Maximum Principle implies that ψ_k is continuous, so $p_k(t)$ must also be continuous. □

Corollary 5.2.1. *Controls are continuous functions of time.*

Since states, co-states, and prices are all continuous, the corollary immediately follows from the continuity of the differential equations characterizing optimal controls.

Lemma 5.3. *Given the set of match surpluses for all types, $\{S(t;j)\}_{j \in J}$ and aggregate leasing, $\mathcal{U}(t)$, there exists a unique quantity of aggregate-searching $\mathcal{V}(t)$ that satisfies operators' optimality and the definitions of aggregate match-flows.*

This constructive proof is given in Section B.3.

5.2.5 Equilibrium equations

The equilibrium is characterized as the solution to a boundary value problem of a system of $4|J| + 3$ first-order ODEs (four per firm-type, two for oilfield services, and one for landowners). These are listed along with market-clearing conditions and transversality conditions in Appendix A on page 68. The system of equations implies a set of stopping times $\{T_j\}_{j \in J}$ for each firm at which the firm stops drilling, $w(t;j) = 0$, and a time T_0 when unleased acreage drops to the threshold where matching is not possible because $\mathcal{U}(t) = \underline{u} \forall t \geq T_0$. The ODEs are all Lipschitz-continuous as long as $K_0 > 0$, so the Picard-Lindelöf theorem guarantees existence and uniqueness of the path of the economy given a set of starting or ending values for the states, co-states, and stopping times. However, as mentioned previously, we have a boundary value problem, not an IVP. The presence of non-sellers implies that the ODEs are not monotonic in the states. Were they to be, uniqueness of the solution to the BVP (though not existence) would follow automatically. Though I have not been able to show uniqueness of the equilibrium path analytically, numerical results suggest that the path is, in fact, unique.

6 Analysis of equilibrium

In this section, we make additional assumptions on the specific functional forms that allow for further analysis of the equilibrium path. We first examine match surplus and lease prices. Then we examine the leasing market to understand the evolution of market tightness, match surplus, and lease prices. This provides some insight into when we might expect lease prices to rise over time. Next, we turn to the drilling market. We show that drilling peaks before the capital stock does if learning effects are not too strong; that the inclusion of an oilfield services sector that must accumulate long-lived capital can smooth out the boom–bust cycle; and that heterogeneity in firms' IP rates and costs may affect firms' drilling rates and the ordering of firms' depletion times differently. We discuss intuition for the time-path of prices for declining drilling services prices and a declining rate of growth in the capital stock. Finally, we examine a numerical simulation of an unexpected 50% price drop analogous to the one in 2014 that affected the Eagle Ford shale. Throughout the discussion, I show numerical simulations to illustrate the current topic of discussion.

6.1 Leasing market

Match surplus and lease-prices Recall that firm-specific match surplus is the difference between buyers' and sellers values: $S(j) = \psi_a(j) - V^u$. Using the co-state equation (36) and the transition equation for V^u, equation (13), we can see that type-specific surplus evolves according to

$$\dot{S}(j) = p_c c_A(j) - \lambda(j) + \tau\theta q(\theta)\bar{S} + \rho S(j).$$

Assumption 5.1 implies that $\lambda = 0$. Consider the case when costs do not change with the quantity of acreage leased, e.g., $c_A = 0$. Under Assumption 5.3, this situation is associated with physical depletion. In this case, firms' valuation of leases, $\psi_a(t; j)$ will rise at the rate of interest to $\psi_{aT}(j)$. However, landowners' opportunity cost grows slower than the rate of interest and must eventually decline to zero as shown in Section 4.2. The gap between firms' valuations and landowners' opportunity cost of leasing their minerals will strictly increase over time, and it will do so faster than the rate of interest. On the other hand, if costs do change with leasing, then $p_c c_A \leq 0$, and match surplus can be growing or shrinking. In fact, if we have economic depletion, making $\psi_{aT}(j) = -\psi_{wT}(j) = 0$, the price could, in fact, fall close to zero before leasing stops at $t = T_0$ if leasing continues almost until the point when firms stop drilling.

We can re-write equation (11) to show that the price of mineral leases is a weighted average of firms' and landowners values: $p_a(j) = \tau\psi_a(j) + (1-\tau)V^u$. Whatever changes firms' lease valuations, ψ_a, changes p_a directly through the first term and also indirectly through the transition equation of the second. Because firms' shadow-value for leases is based on the profitability of all of their future extraction, initial landowners still recieve the benefits of all future technolical progress as long as it is expected.

Because landowners' opportunity cost always grows slower than the rate of interest and firms' value for leases grows no faster than the rate of interest, prices of mineral rights must grow strictly slower than the rate of interest regardless of the type of depletion (since $c_A \leq 0$). This evident in the equation for $\dot{p}_a$:

$$\frac{\dot{p}_a(j)}{p_a(j)} = -\tau\frac{-c_A(j) + \lambda(j) + (1-\tau)\theta q(\theta)\bar{S}}{p_a(j)} + \rho.$$

The equation implies that p_a will shrink in present-value terms. Thus, barring unexpected changes, landowners who sell at $t = 0$ recieve higher compensation than at any other landowner regardless of whether depletion is physical or economic and regardless of future technological progress. This suggests that any "sellers remorse" on the part of Eagle Ford landowners who sold early for lower prices is primarily due to unexpected changes in their beliefs about the future. Even though p_a must decrease in present-value terms, it can can still rise in current value terms. A maximum in $p_a(t)$ is, in fact, guaranteed if V^u is rising at some point. Additionally, a maximum in V^u will always pre-date a maximum in $p_a(j)$ (at least under physical depletion) since firms' lease-valuations can still be climbing ($\dot{\psi}_a(t) > 0$) when when landowners' value crests and $\dot{V}^u(t) = 0$.

Market tightness

Assumption 6.1. *Firms' search-cost function is quadratic:* $\kappa(v) = \frac{\kappa}{2}v^2$.

A quadratic search cost allows for a convenient, closed-form characterization of optimal searching, $v(t;j)$. Under Assumption 6.1, the FOC for optimal leasing, equation (43), implies that

$$v(j) = \frac{1-\tau}{\kappa} q(\theta)\sigma \max\{S(j), 0\}$$

Assumption 6.2. *The aggregate match function is $M(\mathcal{U},\mathcal{V}) = \overline{m}\mathcal{U}^{\overline{\mu}}\mathcal{V}^{1-\overline{\mu}}$, and $\overline{\mu} \in (0,1)$.*

The second assumption, when combined with Assumption 5.1 that $A(t;j) - W(t;j) > 0 \forall t < T_0$, means we can solve for closed-form equations that characterize equilibrium dynamics in the leasing market. We can write match-rates for firms and landowners as $q(\theta) = \overline{m}\theta^{-\overline{\mu}}$ and $\theta q(\theta) = \overline{m}\theta^{1-\overline{\mu}}$, respectively. Define $\mu = \frac{1-\overline{\mu}}{1+\overline{\mu}}$.[26] After substituting $q(\theta)$ into the definition of aggregate searching, equation (4), using the definition of expected match surplus, equation (12), we obtain a closed-form expression for aggregate searching in terms of state variables:

$$\mathcal{V} = \mathbb{1}\left[\mathcal{U} > \underline{u}\right]\mathcal{U}\left[\left(\frac{\overline{m}(1-\tau)}{\kappa}\right)\frac{\sigma\bar{S}}{\mathcal{U}/n}\right]^{\frac{1+\mu}{2}}. \tag{48}$$

As long as $\mathcal{U} > \underline{u}$, aggregate searching, $\mathcal{V}$, strictly increases with unleased acreage, $\mathcal{U}$, as well as aggregate match surplus, $\bar{S}$. The definition of market tightness, equation (6), implies that

$$\theta = \mathbb{1}\left[\mathcal{U} > \underline{u}\right]\left[\left(\frac{\overline{m}(1-\tau)}{\kappa}\right)\frac{\sigma\bar{S}}{\mathcal{U}/n}\right]^{\frac{1+\mu}{2}}. \tag{49}$$

Just as firms' co-states reflect future paths, not contemporaneous state values, future market tightness, not current market tightness, will tend to increase landowners' opportunity cost of selling, V^u, and, hence, the price landowners receive and their share of the firms' lease-valuation. We can see this most easily seen by substituting landowners' match

[26] The following facts are useful:

$$\mu = \frac{1-\overline{\mu}}{1+\overline{\mu}} \implies \mu + \mu\overline{\mu} = 1-\overline{\mu} \implies \overline{\mu}(1+\mu) = 1-\mu \implies \overline{\mu} = \frac{1-\mu}{1+\mu}$$

Also note that

$$1+\mu = 1 + \frac{1-\overline{\mu}}{1+\overline{\mu}} = \frac{(1+\overline{\mu})+(1-\overline{\mu})}{1+\overline{\mu}} = \frac{2}{1+\overline{\mu}}$$

rates into equation (15) and focusing on its definition during the period of active leasing ($t \in [0, T_0]$):

$$V^u(t) = e^{\rho t} \int_t^{T_0} e^{-\rho s} \left(\tau \overline{m} \theta^{1-\overline{\mu}} \bar{S} \right) ds.$$

We can also see how large match surpluses in the future, $\bar{S}$, will tend to increase the current value of V^u. Of course, when V^u is large, $\bar{S}$ must also be smaller. Intuitively, this means that early landowners have a high opportunity cost of leasing because the subsequent landowners must be capturing a high share of match surpluses.

Taking a growth rate of equation 49 yields that

$$\begin{aligned} \left(\frac{2}{1+\mu} \right) \frac{\dot{\theta}}{\theta} &= \frac{\dot{\sigma}}{\sigma} + \frac{\dot{\bar{\bar{S}}}}{\bar{\bar{S}}} - \frac{\dot{\mathcal{U}}}{\mathcal{U}} \\ &= \frac{\dot{\bar{\bar{S}}}}{\bar{\bar{S}}} + \frac{-\dot{\mathcal{U}}}{\mathcal{U}} \left(\frac{\mathcal{U} - 2\mathcal{A}^n}{\mathcal{U} - \mathcal{A}^n} \right). \end{aligned}$$

As the equation makes clear, the growth rate of market tightness depends on the growth rate of aggregate surplus and the (negative) growth rate of unleased acreage. When $c_A = 0$, match surplus increases faster than the rate of interest during active leasing. This will tend increase market tightness since firms are eager to capture an increasing share of the resource rents.

The rate of decrease in the quantity of unleased acreage has an ambiguous effect on market tightness. When most of the shale is unleased so that $\mathcal{U}$ is big, the right-hand term will be positive as well: most matches are with a seller so that $\sigma \approx 1$. However, once non-sellers make up more than 50% of the remaining acreage and $\mathcal{A}^n \geq 0.5\mathcal{U}$, decreasing returns to searching kick in. Now the increasing returns to scale in the (effective) matching function keep firms from searching and drive market-tightness back down. Note that if $c_A = 0$ and there are no non-sellers ($\mathcal{A}^n = 0$), then market tightness increases monotonically. If, in addition, $\underline{u} = 0$, then market tightness will asymptote to infinity.[27]

Aggregate leasing rate Substituting (49) into the FOC for searching, equation 43, and simplifying yields reduced-form equations for type-specific and aggregate equilibrium

[27] Alternatively, finiteness of market tightness is ensured when the aggregate matching function exhibits sufficiently increasing returns to scale in unleased acreage, $\mathcal{U}$.

leasing when $\mathcal{U} > \underline{u}$:

$$\dot{A}(j) = (\overline{m}\sigma)^{1+\mu}\left(\frac{1-\tau}{\kappa}\right)^{\mu}\left(\frac{\mathcal{U}}{\bar{S}n}\right)^{1-\mu}\max\{S(j),0\}$$
$$\dot{\mathcal{A}} = n^{\mu}(\overline{m}\sigma)^{1+\mu}\left(\frac{1-\tau}{\kappa}\right)^{\mu}\mathcal{U}^{1-\mu}\bar{S}^{\mu}.$$

But for the presence of non-sellers, leasing would be a constant returns to scale function of unleased acreage, $\mathcal{U}$, and the aggregate match surplus, $\bar{S}$. However, equilibrium leasing exhibits increasing returns to scale in $\mathcal{U}$ because of σ. The equation for $\dot{\mathcal{A}}$ shows how as available acreage is leased, decreases in $\mathcal{U}$ and σ depress leasing, despite a possibly increasing surplus. This is even clearer when we examine the rate of change in aggregate leasing:

$$\frac{\ddot{\mathcal{A}}}{\dot{\mathcal{A}}} = (1-\mu)\frac{\dot{\mathcal{U}}}{\mathcal{U}}\left(\frac{\mathcal{U}+\frac{2\mu}{1-\mu}\mathcal{A}^n}{\mathcal{U}-\mathcal{A}^n}\right) + \mu\frac{\dot{\bar{S}}}{\bar{S}}.$$

Unleased acreage is always non-increasing, e.g., $\dot{\mathcal{U}} \leq 0$, which provides a drag on aggregate leasing, $\dot{\mathcal{A}}$. The term $\left(\mathcal{U}+\frac{2\mu}{1-\mu}\mathcal{A}^n\right)/\left(\mathcal{U}-\mathcal{A}^n\right)$ is due to σ, and when the quantity of unleased acreage is large at $t=0$, it is closer to 1. As $\mathcal{U} \to \mathcal{A}^n$, however, the term goes to ∞, driving $\dot{\mathcal{A}}$ to zero as acreage is depleted.

Simulation To illustrate these dynamics, I simulate the equilibrium path of a shale boom with physical depletion of resources, oilfield services firms, no learning or knowledge spillovers, and a unit measure of homogenous operators.[28] Figure 11 shows the paths of the stocks and flows of leasing and drilling (A and W in the top pane, and w and $\dot{A}$ in the bottom). The dashed, vertical line marks the time $t = T_0$ at which leasing stops because we reach the minimum unleased acreage threshold and the leasing market shuts down when $\mathcal{U}(t) = \underline{u}$. The plot shows that firms race to accumulate a leasing position, even though they wait a significant amount of time before drilling. Because firms are atomistic, they

[28] Unless otherwise specified, all numerical simulations use a version of the model that eliminates any learning or heterogeneity, includes oilfield services, and sets the mass of firms to one. Simulations are done using the method of reverse shooting (Judd, 1998, Ch. 10). This involves searching over terminal stocks of leasing and capital as well as boundary times T_0 and all but one of the terminal drilling times $\{T_j\}_{J\{1\}}$ such that stocks all hit their starting values. I used MATLAB's `ode45` solver to integrate the ODEs backwards from $t = T$ to $t = 0$. Due to the nonlinearity of the model, tight integration tolerances (10^{-12}) were required. Though the simulations were quite sensitive to starting values, I was able to hit all target initial conditions within 10^{-5}.

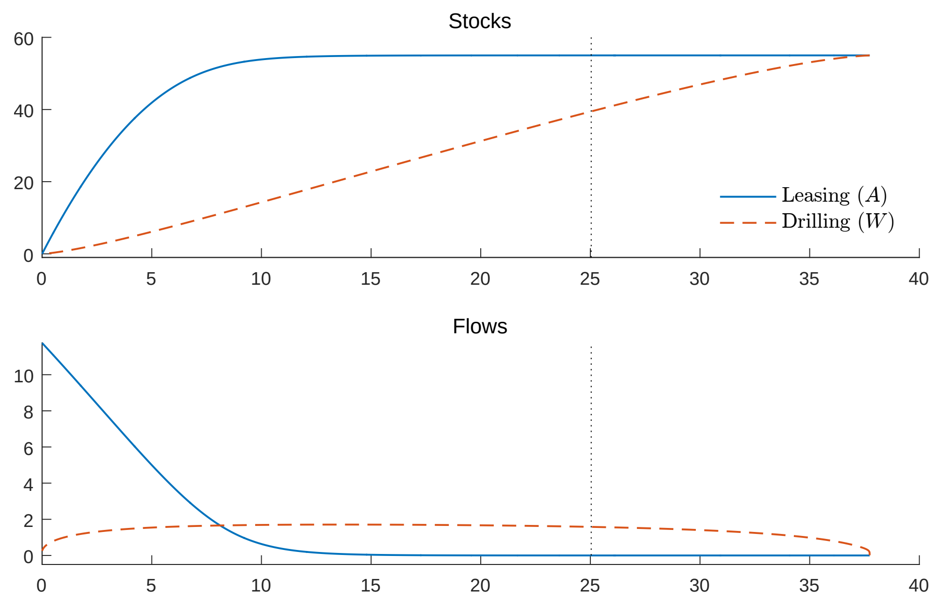


Figure 11: Leasing vs drilling (physical exhaustion)

can individually acquire any quantity of leases before $t = T_0$; however, it is not optimal to delay leasing to match drilling because time-consuming searching in a tight market would be very costly. The high rates of leasing compared with low rates of drilling resemble this qualitative feature of actual Eagle Ford leasing and drilling rates shown in Figure 5a.

Figure 12 focuses on the details of the leasing market. The top panel shows the variables relevant to the bargaining problem: firms' valuations, ψ_a, landowners' opportunity cost, V^u, and the price, p_a. The three are quite close together at the outset, growing at or near the discount rate, ρ. During this initial period, landowners appear to be capturing a large share of firms' lease valuations. The slower (and eventually negative) growth in landowners' opportunity cost is due to the fact that the passage of time reduces a landowners' likelihood of capturing a high share of firms' values. The difference between ψ_a and V^u, the match surplus, increases monotonically. A widening surplus and decreasing quantity of unleased acres first leads to increasing market-tightness. When we view firms' and landowners valuations in present-value terms (the second pane), we can see that firms' valuations are constant. This is the leasing-counterpart to the Hotelling rule for optimal extraction. In contrast, negotiated lease prices and landowners' opportunity cost of leasing are strictly falling, despite having a peak in current-value terms. This illustrates how landowners who

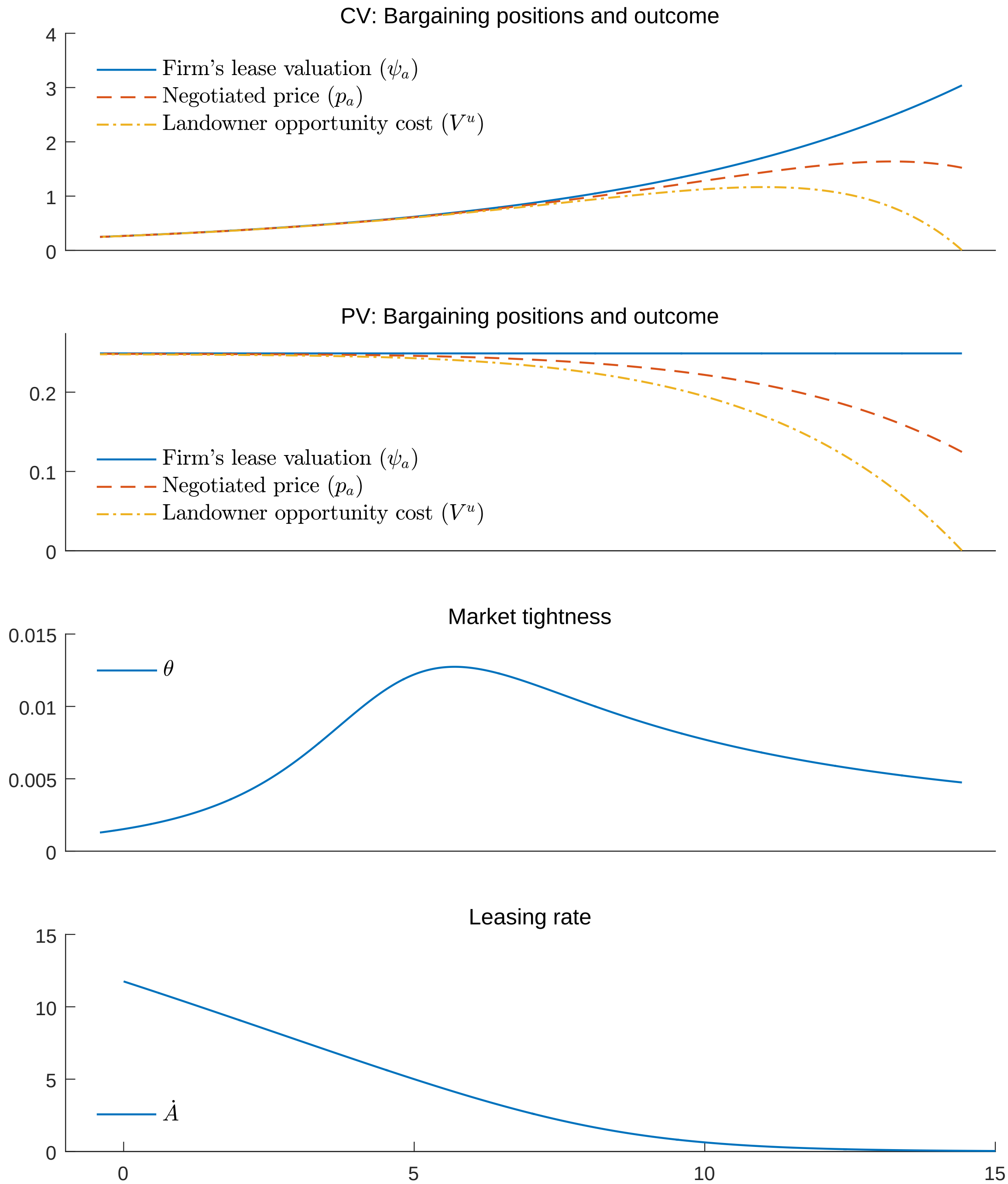


Figure 12: Lease-market (physical depletion)

lease at the beginning of a boom obtain the highest share of rents (barring unexpected technological improvements). The return to searching for firms decreases as firms find it increasingly hard to locate selling landowners. This causes the rate of searching to drop faster than the rate at which unleased acreage shrinks, and market tightness falls. Once $\mathcal{U}$ reaches $\underline{u}$, matching stops, and leasing market shuts down. Finally, the aggregate leasing rate, $\dot{\mathcal{A}}$, strictly falls over time.

6.2 Analysis of drilling

As explained earlier, the link between the value of leases and firms' resource extraction problem is solely via the path of the co-state $\psi_a(t;j)$ regardless of whether depletion is physical or economic. Furthermore, $\psi_a(t;j)$ is entirely determined by the terminal co-state $\psi_{wT}(j)$, the time of depletion, T_j, and the decrease in drilling costs due to the marginal lease purchase, $p_c c_A$. The price of drilling services affects landowners because it changes firms' optimal extraction rate (and therefore the depletion time) as well as and cost reductions from the marginal lease if having a larger lease portfolio reduces costs. Capital markets can affect landowners via the price of drilling services.

Assumptions on drilling costs We now make some assumptions on the functional forms of operators' drilling costs and oilfield services production function that allow for further characterization of the equilibrium path of drilling. Assumption 6.3 puts more structure on the way state variables affect drilling costs. It allows us to separate out the effects of heterogeneity, depletion, learning-by-doing, and knowledge spillovers while remaining flexible. Similarly, Assumption 6.4 imposes a Cobb-Douglas production function that allows for a convenient characterization of the price of oilfield services.

Assumption 6.3. *The drilling cost function has the form*

$$c(w, A, W, \mathcal{W}; j) = \varphi(j)E(w)G(A, W)D(W)\mathcal{D}(\mathcal{W}). \tag{50}$$

Also, $0 < \varphi(j) < \infty \forall j$, *and the component functions have the following characteristics to ensure the convexity of* $c(\cdot)$*:* $E(0) = 0$, $E', E'' > 0$, *and* $\lim_{w\to\infty} E'(w) = \infty$*;* $G_A, G_{AW} \leq 0$, $G_W, G_{AA}, G_{WW} \geq 0$, *and* $G(A, 0) = 1 \forall A > 0$*;* $D(0), \mathcal{D}(0) \geq 1$, $\lim_{W\to\infty} D(W) = \lim_{\mathcal{W}\to\infty} \mathcal{D}(\mathcal{W}) = 1$, $D', \mathcal{D}' \leq 0$, *and* $D'', \mathcal{D}'' \geq 0$.

There are several components to the drilling cost function. First, the parameter $\varphi(j)$

is a *type-specific productivity* parameter. Second, $E(w)$ is a convex function characterizing base ***e**xtraction* costs. Third, $G(A, W)$ is a function that may characterize ***g**eological* constraints that cause costs to increase as a larger fraction of the firm's acreage is drilled. This term is meant to capture economic depletion effects. Fourth, $D(W)$ captures private efficiency gains from *learning-by-**d**oing*, and finally, $\mathcal{D}(\mathcal{W})$ captures industry-wide *knowledge spillovers* (like an industry-wide learning-by-doing effect). Though the functions $D(\cdot)$ and $\mathcal{D}(\cdot)$ may be identical, only the first will directly affect the firms' opportunity cost of drilling and be internalized; the second will not.

It will be convenient to define a new time-varying state variable capturing the effect of state variables and type on drilling costs:

$$N(t;j) \equiv \varphi(j) \times G(A(t;j), W(t;j)) \times D(W(t;j)) \times \mathcal{D}(\mathcal{W}(t)).$$

Now we can write demand for drilling services in terms of the control and a single state variable: $c^d(t;j) = E(w(t;j))N(t;j)$. The FOC for optimal drilling, equation (42), becomes quite simple once time and type-subscripts are dropped:

$$\widehat{p} \leq p_c E'(w)N + (-\psi_w).$$

Drilling behavior around depletion We are now in a position to make some qualitative characterizations of drilling rates and profits immediately before depletion occurs.

Lemma 6.1. *Drilling is strictly declining before depletion:* $\exists T_j^d < T_j$ *such that* $\dot{w}(t;j) < 0 \forall t > T_j^d$. *Also, resource rents are strictly positive before depletion:* $\exists T_j^r < T_j$ *such that* $-\psi_w(t;j) > 0 \forall t \in (T_j^r, T_j)$.

See Appendix B.3 for a proof of Lemma 6.1. We can use the FOC for drilling to see that the impacts of depletion and learning cannot be too strong as the firm depletes its resource base. Differentiating the FOC for drilling during periods in which $w > 0$ implies that

$$\frac{\dot{\psi}_w}{p_c} - \frac{\dot{p}_c}{p_c} c_w - c_{wW}\dot{W} + \underbrace{\left(-c_{wA}\dot{A}\right) + \left(-c_{w\mathcal{W}}\dot{\mathcal{W}}\right)}_{\geq 0} = \underbrace{c_{ww}}_{>0}\ddot{W}.$$

When drilling is decelerating ($\ddot{W} < 0$) during $t \in (T_j^d, T_j)$, we require that

$$\frac{\dot{\psi}_w}{p_c} - \frac{\dot{p}_c}{p_c} c_w - c_{wW}\dot{W} < 0.$$

If the price of services is decreasing (the normal case) because the demand for services decreases faster than the capital stock as drilling declines, this would further imply that $\frac{\dot{\psi}_w}{p_c} - c_{wW}\dot{W} < 0$. We can use this fact to bound both learning-by-doing and depletion effects. The increase of drilling services requirements with respect to an additional well is c_W. It can be either positive or negative depending on whether learning dominates ($c_W < 0$) or depletion dominates ($c_W > 0$). Should we have that $\dot{\psi}_w \geq 0$, which corresponds to economic depletion, then it must be that $c_{wW} = E'[G_W D + GD']\mathcal{D} > 0$ and depletion dominates learning, e.g., $G_W D > -GD'$. Should we have physical depletion so that $\dot{\psi}_w < 0$, then we can bound c_W. First $\dot{\psi}_w < 0 \implies c_W < -\rho\psi_w$. Since $-\rho\psi_w > 0$, this means that c_W cannot be too positive, i.e., depletion effects cannot be too strong. Second, c_W cannot be too negative. Since $c_W = c_{wW}\frac{E}{E'}$, we can write that $\frac{\dot{\psi}_w}{p_c}\left(\frac{E'}{E}\dot{W}\right)^{-1} < c_W$. This implies that learning effects cannot be too strong; otherwise, drilling would accelerate.

Impact of learning Extracting oil and gas from shale was made possible by dramatic reductions in the cost of extraction. The initial shale wells drilled by shale pioneer George Mitchell were experimental, and some likely resulted in negative accounting profits. Nevertheless, these wells were economically profitable since the experience they provided allowed Mitchell to lower the cost of future wells. Firms have continued to innovate, experiment, and reduce the cost per unit of resource extracted. As discussed in Section 2.3, a number of researchers have found that much of firm learning is a private process, but some knowledge spillovers are also likely present.

The lemma below shows that, all else equal, private learning accelerates depletion more than public knowledge spillovers. This is because firms internalize the present-value of future cost-reductions when we have learning-by-doing, but not when there are knowledge spillovers. A lower opportunity cost means it is optimal to drill faster, and accumulated learning further lowers extraction costs. These two effects both serve to accelerate the time to depletion. The subsequent discussion handles a “George Mitchel” scenario. It shows that only private learning can lead firms to drill initial, unprofitable wells and that these unprofitable wells only happen at the beginning of the boom before drilling rates crest.

Lemma 6.2. *Suppose that we have no capital, a unit-mass of homogeneous firms, and physical depletion:* $p_c = n = G(A, W) = 1$. *Now consider two firms that differ only in whether technological progress is through internalized learning-by-doing or industry-wide knowledge spillovers. Then the firm which internalizes its costs starts drilling faster and depletes its resource first.*

The proof of Lemma 6.2 is in Appendix B.3. The fact that the two firms differ only in whether they internalize the cost reductions from learning implies that their terminal rents, $-\psi_{wT}$ are identical. Thus, the only difference in firms' lease-valuations, $\psi_a(t)$, will be due to their time of depletion. Since the firm which internalizes learning finishes first (shown in the lemma), its lease-valuations are higher. This implies that the landowners in this world receive a higher price for their acreage. It will also translate into a tighter leasing market. Note that if n is large so that industry-wide experience accumulates very quickly, the knowledge-spillover firms may finish drilling first, and the lemma may not hold.

Lemma 6.3. *For drilling unprofitable wells to be optimal, learning-by-doing must have a private component*

The proof of Lemma 6.3 is in Appendix B.3. Intuitively, learning through purely public knowledge spillovers can never induce firms to drill unprofitable, experimental wells to accelerate learning because they do not internalize the economic benefits of doing so.

Lemma 6.4. *Suppose that firms find it optimal to drill unprofitable wells for some period of time, and that* $p_c \geq 1$ *and* $\dot{p}_c \leq 0$. *Then* $-GD' > 0$ *and* $\dot{w} > 0$.

First, this requires that technological progress is, in large part, through private learning-by-doing. Second, learning effects must be dominating depletion effects: ich is what we would expect at the beginning of a shale boom as oilfield services firms accumulate capital,

Proof. Unprofitable wells imply that $\psi_w \geq 0$. Lemma 6.1 implies that there exists some time $T_j^r < T_j$ such that $\psi_w < 0 \forall t > T_j^r$. Continuity of ψ_w and the existence of unprofitable drilling implies that $\psi_w(T_j^r; j) = 0$. Continuity of p_c, the fact that $c(\cdot)$ is twice differentiable, and the co-state transition, equation (35), imply that we must have $c_W < 0$. Assumption 5.3 implies that $c_W < 0 \iff -GD' > G_W D \geq 0$. Differentiate the FOC for drilling, equation (42), to get

$$-\frac{\dot{p}_c}{p_c}c_w + \underbrace{\frac{c_W}{E}}_{<0}\left[\frac{E}{p_c} - E'\dot{W}\right] + \underbrace{\left(-c_{wA}\dot{A}\right) + \left(-c_{w\mathcal{W}}\dot{\mathcal{W}}\right)}_{\geq 0} = c_{ww}\ddot{W}.$$

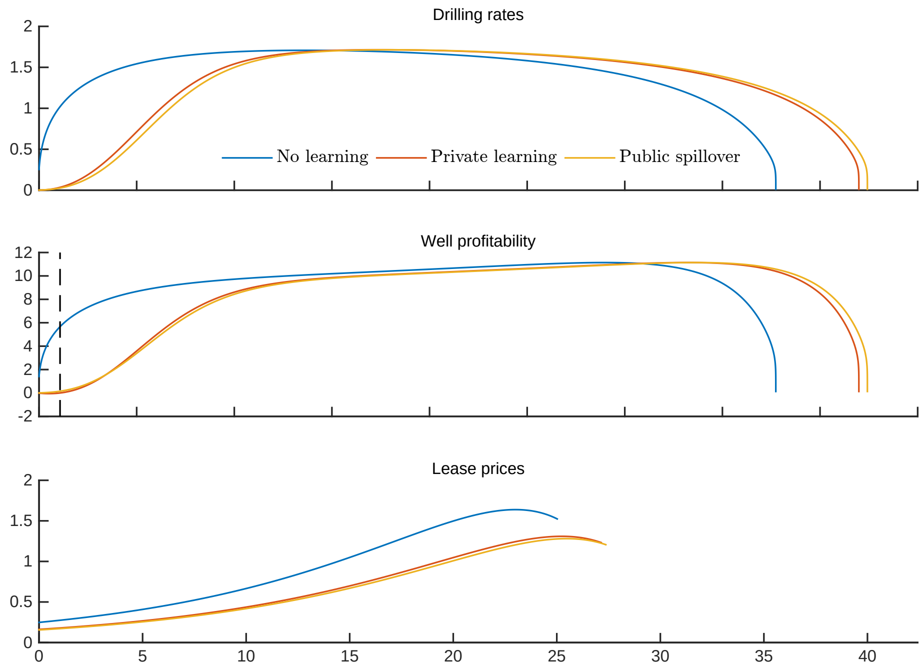


Figure 13: Simulation: no learning, purely private, and purely public learning

If prices are high ($p_c(T_j^r) \geq 1$) but not rising ($\dot{p}_c(T_j^r) \leq 0$), then the assumption that $E(0) = 0$ plus convexity implies that $0 \geq E(\dot{W}) + E'(\dot{W})(0 - \dot{W})$. Then c_{ww} from Assumption 5.3 implies $\ddot{W} > 0$. □

Figure 13 shows numerical simulations of three different scenarios: no learning, purely private learning-by-doing, and public knowledge spillovers.[29] The dashed line in the middle pane marks the time at which profits from drilling turn positive in the private learning world. The need to learn slows down initial drilling, which helps smooth the initial ramp-up in drilling and helps the model match the qualitative dynamics we saw in Eagle Ford drilling. Additionally, we can see that private learning accelerates drilling and depletion, and this raises lease prices modestly.

[29]Since learning is modeled as a process of reducing cost inflators $D(W)$ and $\mathcal{D}(\mathcal{W})$, the costs in the no-learning scenario are somewhat lower as $D(W) = \mathcal{D}(\mathcal{W})$ in that case.

6.3 Oilfield services and capital

In Section 6.3, we consider the oilfield services sector and the role of long-lived capital. First, we will see that in a simple model with no learning, drilling hits its maximum at $t = 0$ and then declines monotonically. This clearly did not happen in the Eagle Ford shale, as shown in Figure 5a. Instead, drilling has ramped-up gradually. Including an oilfield services sector that must accumulate long-lived, inelastically-supplied capital can help smooth out the boom-and-bust cycle in a depletable resource model. Like learning effects, this allows the model to better match the qualitative dynamics of Eagle Ford drilling.

I parameterize the inverse supply curve in a simple manner that allows us to differentiate the effect of the level and elasticity of the price of capital. A more elastic capital market will translate into lower drilling costs, faster depletion, and higher lease prices. Operators in the Eagle Ford shale benefited from (1) pre-existing midstream infrastructure (capital) that lowered the cost of transporting resources to the coast, and (2) an elastic supply of drilling capital, since this could be constructed or moved from other parts of the country relatively easily. Both factors have increased the profitability of shale. The feasibility of shale development in other countries like Mexico, Argentina, and China will be greatly influenced by the availability of the capital and transportation infrastructure required for shale development.

Assumption 6.4. *The drilling services production function is* $f(L,K) = L^{1-\alpha}K^{\alpha}$, *and* $\alpha \in (0,1)$.

Assumption 6.4 implies that labor demand and its derivatives are:

$$L = \left(\frac{c^s}{K^\alpha}\right)^{\frac{1}{1-\alpha}} \qquad \frac{\partial L}{\partial c^s} = \frac{1}{1-\alpha}\left(\frac{c^s}{K}\right)^{\frac{\alpha}{1-\alpha}} \qquad \frac{\partial L}{\partial K} = -\frac{\alpha}{1-\alpha}\left(\frac{c^s}{K}\right)^{\frac{1}{1-\alpha}}.$$

The FOC for labor demand by oilfield services, equation (45), and the capital co-state equation, equation (47), imply that these two equations are

$$p_c = \frac{p_l}{1-\alpha}\left(\frac{c^s}{K}\right)^{\frac{\alpha}{1-\alpha}} \tag{51}$$

$$\dot{\psi}_k = -p_l\frac{\alpha}{1-\alpha}\left(\frac{c^s}{K}\right)^{\frac{1}{1-\alpha}} + (\rho+\delta_k)\psi_k. \tag{52}$$

Assumption 6.5. *The inverse supply curve of capital is* $p_k(\mathcal{I}) = \bar{p}_k \exp\{\mathcal{I}/\eta\}$ *where* $\infty > \bar{p}_k, \eta > 0$.

The assumption on the supply curve of capital breaks its price into two components: a base price, $\bar{p}_k$, which corresponds to zero capital purchases, and a price elasticity, $1/\eta$. The FOC for oilfield services capital investment, equation (46), implies that capital is demanded until its price equals firms' value of capital: $p_k(\mathcal{I}) = \psi_k$. Market clearing, equation (30), implies that investment is $i = p_k^{-1}(\psi_k) = \eta \log(\psi_k/\bar{p}_k)$ and that the law of motion for capital is

$$\dot{K} = \eta \log(\psi_k/\bar{p}_k) - \delta_k K.$$

The top panel of Figure 14 shows labor, capital stocks, and production of oilfield services firms. All three increase initially and then fall. The price of drilling services, which is a monotonic transformation of the labor–capital ratio, falls in every period, though it seems to spend a good portion of the shale boom in a moderate range. Falling prices allow operators to sustain higher levels of drilling for longer, even as their costs fall and resource rents rise. The bottom panel shows investment, i, net investment, $i - \delta_k K$, and the change in the labor force, $\dot{L}$. First, note that high levels of investment are necessary to sustain capital stock levels, even when they are declining. Second, and perhaps more striking, is to note the dramatic changes in the size of the labor force in oilfield services. Hiring rates are extremely high near $t = 0$. However, reductions in their labor force begin even before drilling peaks, and continue for well over half of the life of the shale play. These rapid changes are consistent with the strong nation-wide ramp-up in hiring by oilfield services that coincided with the early years of the shale boom (see Figure 8), but they imply that employment in oilfield services could see wide variation in level over the lifetime of a play much more than capital or drilling rates, even absent randomness in the resource price.

Effect of capital on drilling rates To illustrate the effect that oilfield services and its long-lived capital have on equilibrium dynamics, we consider two models where the only difference is the presence of the oilfield services sector. For both, suppose that there are no depletion or learning effects: $c(w, A, W, \mathcal{W}; j) = E(w)$, implying that $G(A, W) = D(W) = \mathcal{D}(\mathcal{W}) = |J| = \varphi(j) = n = 1$. In the model without capital, assume $\bar{p}_c = \frac{1}{T}\int_t^T p_c(s)ds$, where $p_c(t)$ is determined in the model with capital. Since there is no learning, $c_W = 0$. Positive drilling rates require that the drilling co-state is negative, $\psi_w(t) < 0$, which means its derivative is also negative: $\dot{\psi}_w(t) = \rho\psi_w(t) < 0$. Differentiating the drilling FOC with

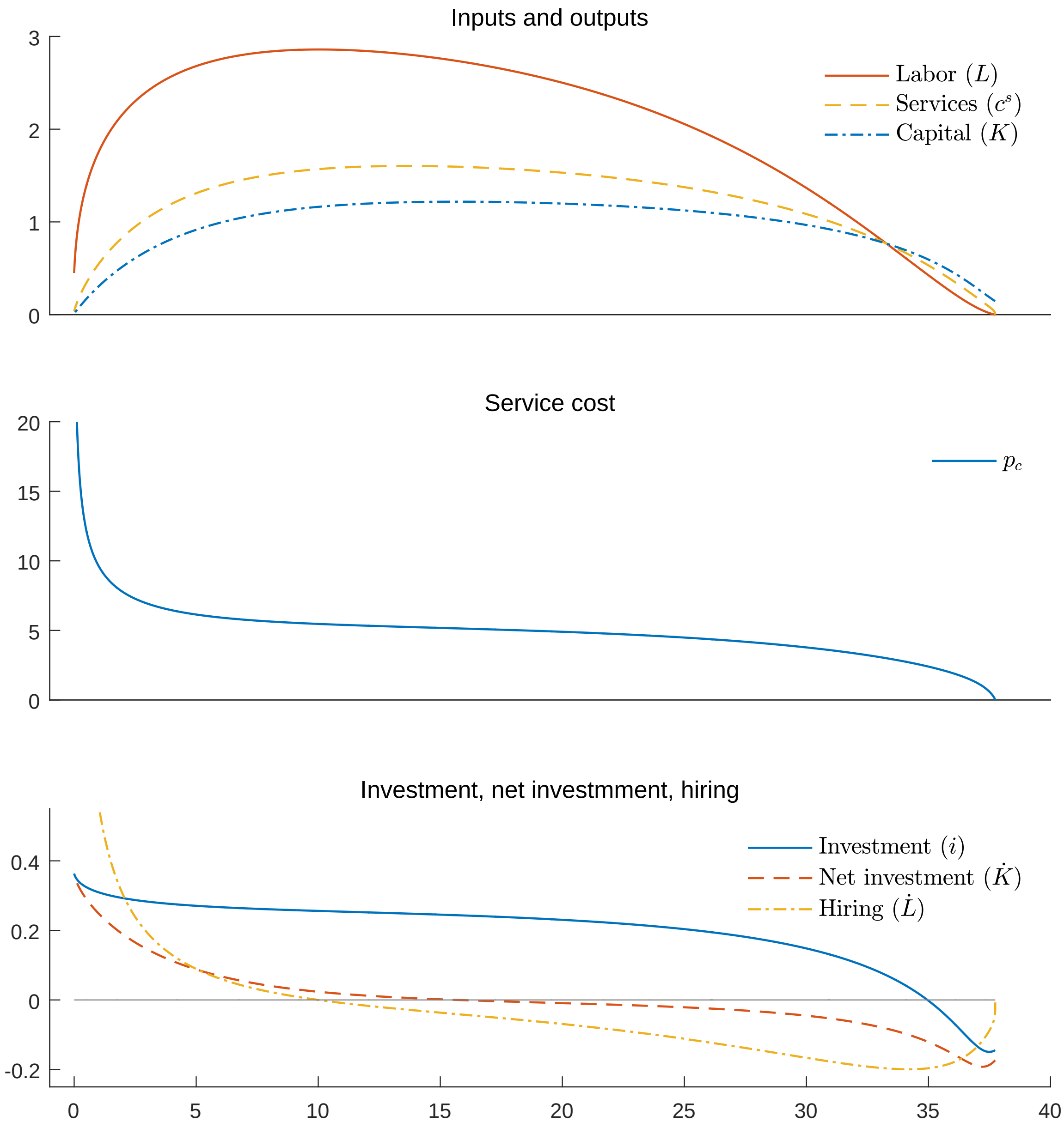


Figure 14: Oilfield services

respect to time implies for any type implies that

$$\dot{\psi}_w = \dot{p}_c E(w) + p_c E'(w)\dot{w} < 0.$$

Without capital $\dot{p}_c = 0$. The fact that drilling is decreasing with time, $\dot{w} < 0$, follows immediately. However, if we allow for an endogenously determined price of services, this is no longer necessarily the case. Differentiating equation (51) with respect to time implies that the change in the price of services is

$$\dot{p}_c = p_c \frac{\alpha}{1-\alpha}\left[\frac{\dot{c}^s}{c^s} - \frac{\dot{K}}{K}\right].$$

Market-clearing for the services industry implies that $c^s = E(w)$ and $\dot{c}^s = E'(w)\dot{w}$. Using this fact, substituting in $\dot{p}_c$, and some algebra yields the following inequality:

$$\dot{\psi}_w = \frac{p_c}{1-\alpha}\left\{E'(w)\dot{w} - \alpha\frac{\dot{K}}{K}\right\} < 0.$$

Now consider $t = 0$. When capital is very small, oilfield services firms will purchase more, making the capital stock growth rate very large. Since the rate of change in the drilling co-state, ψ_w, is finite and negative, a small initial capital stock, K_0, will be associated with an increasing initial drilling rate: $\dot{w}(0) > 0$. This implies that drilling does *not* hit its maximum at $t = 0$, so the presence of capital smooths out the boom. If capital is cheaper ($\bar{p}_k$ is smaller) or the market is more elastic (η is big), we should expect the capital stock to grow faster, and drilling to increase faster. Of course, if there is a large quantity of capital at $t = 0$ and the capital growth rate is either small or negative, then rate of drilling can also be decreasing: $\dot{w}(0) < 0$.

Figure 15 compares the simulation of drilling rates and the price of oilfield services in the baseline model with capital (also shown in Figures 11, 12, and 14) with a model that removes oilfield services and its long-lived capital. The plot shows how high initial prices for oilfield services due to low initial capital stocks slow initial drilling. In contrast, when long-lived capital does not function as a brake, drilling immediately hits its maximum and declines thereafter. Figure 5a shows how drilling in the Eagle Ford has a smooth start; it does not have the meteoric takeoff consistent with the no-capital case.

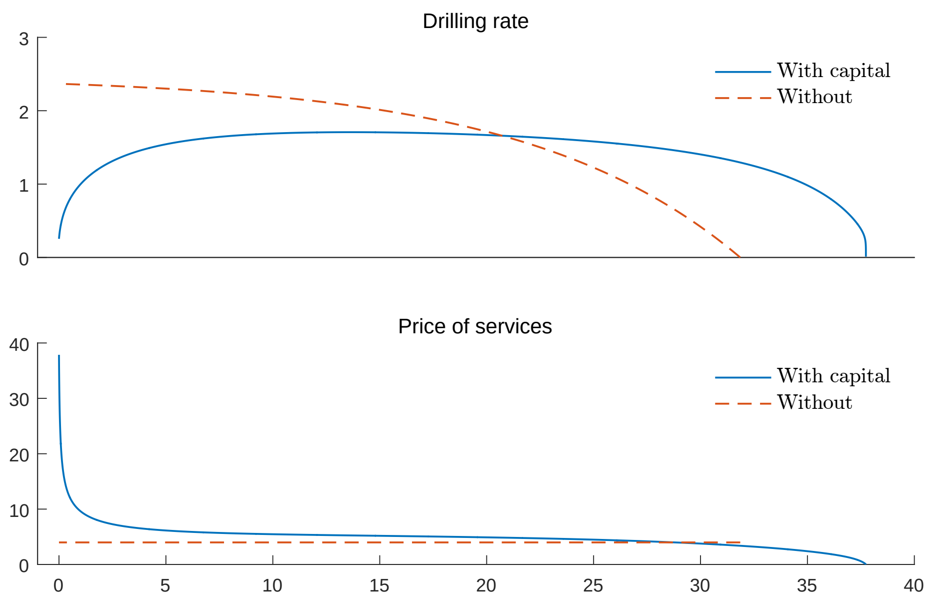


Figure 15: Drilling with and without long-lived capital

6.4 Heterogeneity in firms

To make the model more tractable analytically and computationally when capital and heterogeneity are present, I assume that base extraction costs costs have a very simple functional form and that depletion is physical. I provide some intuition about the qualitative relationship between the paths of aggregate drilling rates and the capital stock. Then, I provide a simple result that shows how heterogeneity in costs affects lease prices and portfolios across firm types.

Assumption 6.6. *Drilling costs are quadratic:* $E(w) = \frac{w^2}{2}$ *and* $G(A, W) = 1$*.*

Assumption 6.6 guarantees that we have physical depletion, and it makes analytical characterization of the equilibrium path straightforward since $E'(w) = w$. Additionally, since the marginal cost of drilling no wells is zero, it means first that $-\psi_{wT}(j) = \widehat{p}(j)$ and second that we can ignore non-negativity constraints on drilling since they never bind. Finally, the term $N(t; j)$ only captures heterogeneity and learning, and it is weakly decreasing over time since $D' \leq 0, \mathcal{D}' \leq 0 \implies \frac{\dot{N}}{N} \leq 0$. Aggregating over firm's optimal drilling (given prices) implies that the price of drilling services, p_c, must satisfy

$$c^d = (p_c)^{-2}\frac{n}{2}\int \frac{(\hat{p}+\psi_{wj})^2}{N(j)}dF(j).$$

As we would expect, demand for services decreases with price. With the particular functional form of $E(\cdot)$, technological change implies N decreases, thereby increasing demand for drilling services. Using market-clearing in the services sector, equation (29), and substituting services demand into the FOC for the service production, equation (51), implies that

$$(p_c)^{\frac{1+\alpha}{\alpha}} = \left(\frac{p_l}{1-\alpha}\right)^{\frac{1-\alpha}{\alpha}}\frac{n}{2}\int \frac{(\hat{p}+\psi_w)^2}{KN(j)}dF(j).$$

The equation shows that increases in the capital stock reduce the price of services, but learning increases the price because it accelerates drilling. The growth rate of p_c will be

$$\frac{\dot{p}_c}{p_c} = \frac{\alpha}{1+\alpha}\frac{\int\left[-\frac{\dot{K}}{K}+2\frac{\dot{\psi}_w(j)}{\widehat{p}(j)+\psi_w(j)}-\frac{\dot{N}(j)}{N(j)}\right]\frac{(\hat{p}+\psi_{wj})^2}{KN(j)}dF(j)}{\int\frac{(\hat{p}+\psi_w)^2}{KN(j)}dF(j)}.$$

The term $\frac{\dot{\psi}_w(j)}{\widehat{p}(j)+\psi_w(j)}$ captures the idea that increasing resource rents (due to $\dot{\psi}_w < 0$) correspond to falling drilling rates and, hence, demand for services. As the opportunity cost of drilling, $-\psi_w(j)$, grows in magnitude to $\widehat{p}(j)$, the numerator is increasing as the denominator goes to zero. This causes the fraction to asymptote to $-\infty$ and drives p_c towards zero. We should expect that the capital stock decreases ($\dot{K} < 0$) as we approach the end of the shale, serving to mitigate the decrease in demand for services on their price. Since capital has positive scrap value ($p_k(-K_T) > 0$), oilfield services firms shouldn't need to reduce ending capital stocks to zero. That means the cessation of drilling effect, $\frac{\dot{\psi}_w(j)}{\widehat{p}(j)+\psi_w(j)}$, can cause the price of services to keep falling even as capital is sold off at the end.

The aggregate drilling rate and its second derivative will be

$$\begin{aligned}\dot{\mathcal{W}} &= \int \frac{\widehat{p}(j)+\psi_w(j)}{p_c N(j)}dF(j)\\ \ddot{\mathcal{W}} &= \int w(j)\left[\frac{\dot{\psi}_w(j)}{\widehat{p}(j)+\psi_w(j)}-\frac{\dot{N}(j)}{N(j)}\right]dF(j)-\dot{\mathcal{W}}\frac{\dot{p}_c}{p_c}.\end{aligned}$$

Under the assumption that learning effects are minor, so that $\dot{N}/N \approx 0$ at any point drilling crests ($\ddot{\mathcal{W}} = 0$), the fact that $\dot{\psi}_w \leq 0$ means that the price of drilling services is still falling ($\dot{p}_c/p_c < 0$). Using the production function and the FOC for labor demand, equation (51), we can see that the price of drilling services is related to the capital-labor ratio as

$$p_c = \frac{p_l}{1-\alpha}\left(\frac{K}{L}\right)^{-\alpha} \implies \frac{\dot{p}_c}{p_c} = -\alpha\frac{(\dot{K/L})}{K/L}.$$

This makes it clear that when drilling crests and learning effects are minor, the capital-labor ratio is still rising. Furthermore, the capital stock must also still be increasing. Nevertheless, if learning effects are very strong, capital may crest before aggregate drilling does.

Lemma 6.5. *Suppose that firms are differentiated in costs but not initial production rates:* $\varphi(j) \neq \varphi(j')\forall j \neq j'$ *and* $q_0(j) = q_0(j')\forall j \neq j'$*. Also suppose that there is no private learning-by-doing:* $D(W) = 1$*. Then the more efficient firms finish drilling faster and have a bigger lease portfolio.*

The proof on Lemma 6.5 in Appendix B relies on the fact that firms' terminal rents must be the same when each physically depletes its resource because Assumption 6.6 means that $E'(0) = 0$. The only way one firm can lease (and drill) more than the other, is if it values the land more highly than the other, which implies it finishes drilling first. This must also be the more efficient firm. If firms differ on the amount they produce but not on costs (and there are no learning or depletion effects), the more efficient firm does not necessarily finish first. In fact, in later simulations, I find that the more productive firm purchases more land and finishes drilling later. These results suggest that increasing well productivity and increasing cost efficiency may have distinct effects on the time-to-depletion.

Figure 16 depicts stocks and flows of leases under heterogeneity in the cost parameter, $\varphi(j)$, and IP rates, $q_0(j)$. These correspond to the left and right columns, respectively. Heterogeneous parameters are set so that their mean is the same as the baseline case. When costs are changed, the low-cost firm does finish drilling slightly earlier, consistent with theory. This requires the firm to drill at a higher rate. The difference in drilling times is only very slight, meaning leasing-valuations are very similar, but this still induces a substantial difference in the size of their total lease-portfolios. When IP rates are different, the more productive firm leases also leases more because its lease-valuations are bigger, and it drills at a higher rate as well. However, the firm also takes longer to deplete its

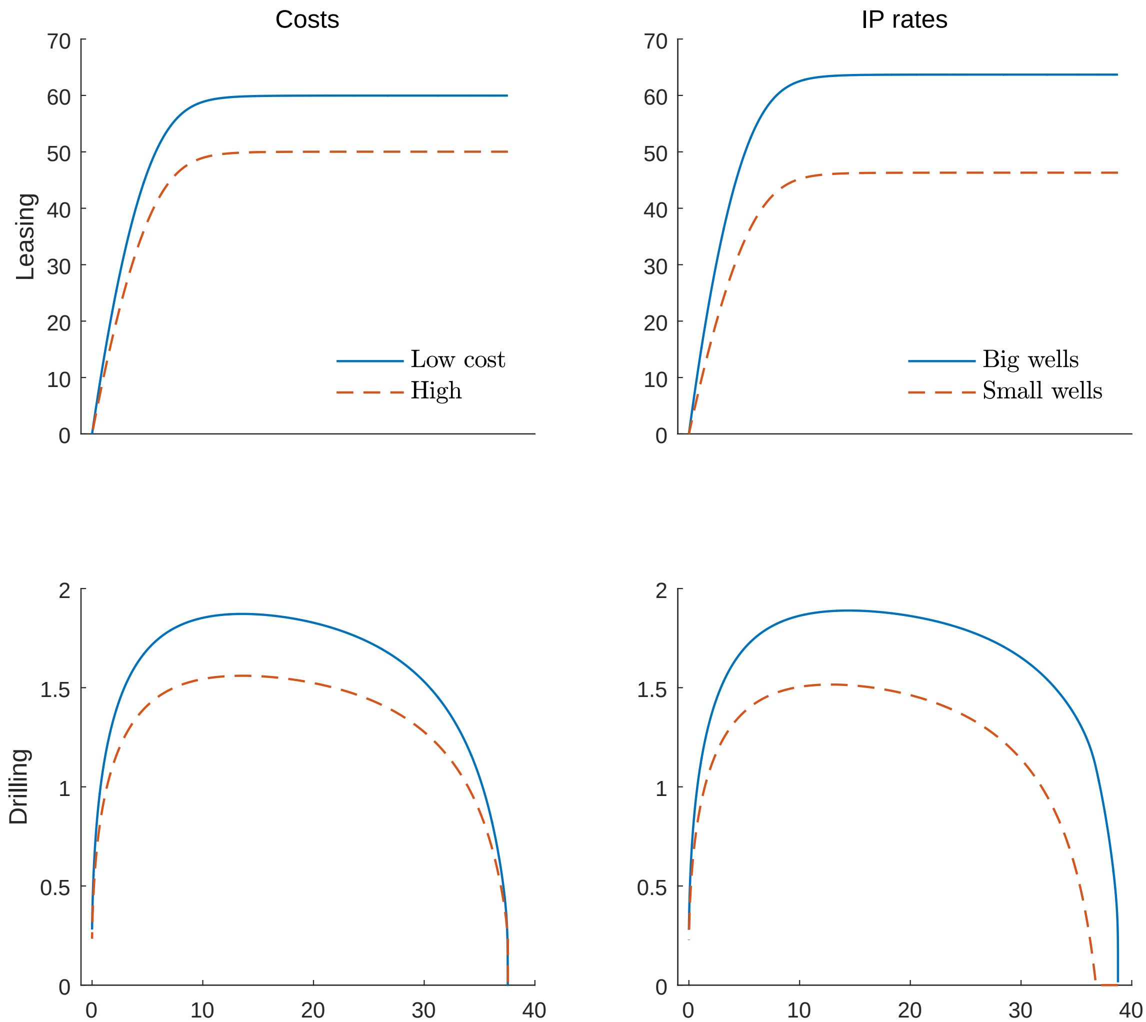


Figure 16: Heterogeneity in cost vs IP rate

stock of leases.

6.5 Competition

I do not provide an analytical characterization of the relationships between the number of firms, entry costs, and the path of the equilibrium. However, simulations show that these relationships tend to behave as expected: more competition increases prices, increases leasing and drilling rates, and decreases the value of being in the market. Part of the increase in lease prices has to do with tighter markets, and part is due to the fact that earlier depletion times raise lease-valuations because terminal valuations are discounted less, in addition to the larger mass of firms in the market. Figure 17 shows lease stocks, lease-prices, aggregate drilling rates, and the value of entering the market for $n = 1, 2, 5$ and, for the last plot only, $n = 10$. We can conclude that, as expected, more competition is good for landowners, makes the shale boom more extreme, and reduces operator profits.

6.6 Simulation of a price drop

Since oil prices dropped more than half in late 2014, Figure 18 simulates an unexpected 50% price drop midway through leasing in a shale. The simulation appears to capture the same qualitative dynamics that we have seen in the Eagle Ford post-2014. In the simulation, lease prices drop significantly because lease valuations do (due to both the fall in $\widehat{p}$ as well as the increase in T_j). They also drop because future market tightness θ, does as well. Though royalty rates do not appear to have dropped dramatically in response to prices in the Eagle Ford (see Figure 4), we should remember that the royalty contracts we see empirically still pass on the price drop directly to landowners.

Leasing and drilling rates do drop in the simulation, but to a much lesser degree than lease prices. The smaller drop in drilling and leasing is due to the fact that both lease prices and oilfield services prices drop. In the simulation, the price of oilfield services drops by about a third, and Figure 9 shows that they dropped a similar amount after the 2014 crash in oil prices. Labor in oilfield services plummets in the simulation, just as Figure 9 shows it did during the 2015–2016 period. In the model simulation, declining drilling rates correspond to operators' increasing marginal productivity. In light of this result, the counter-cyclical increase in wages in the oil extraction sector following the drop in prices shown in Figure 9 is not so surprising.

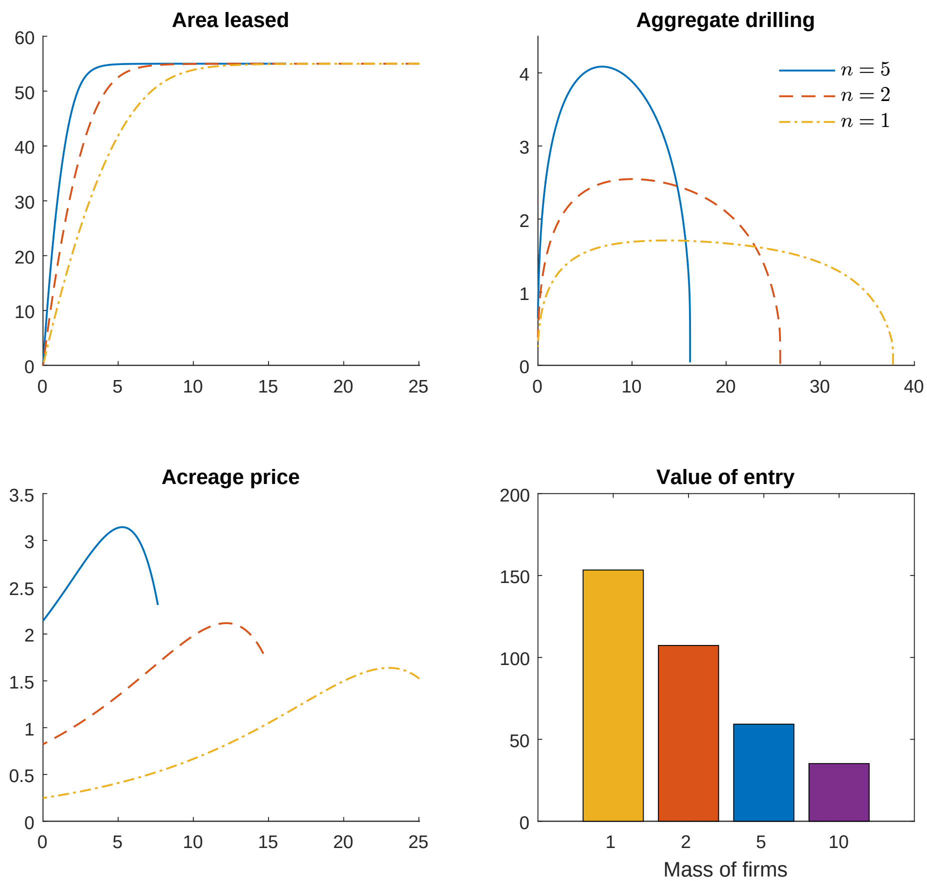


Figure 17: Impact of entry costs on leasing

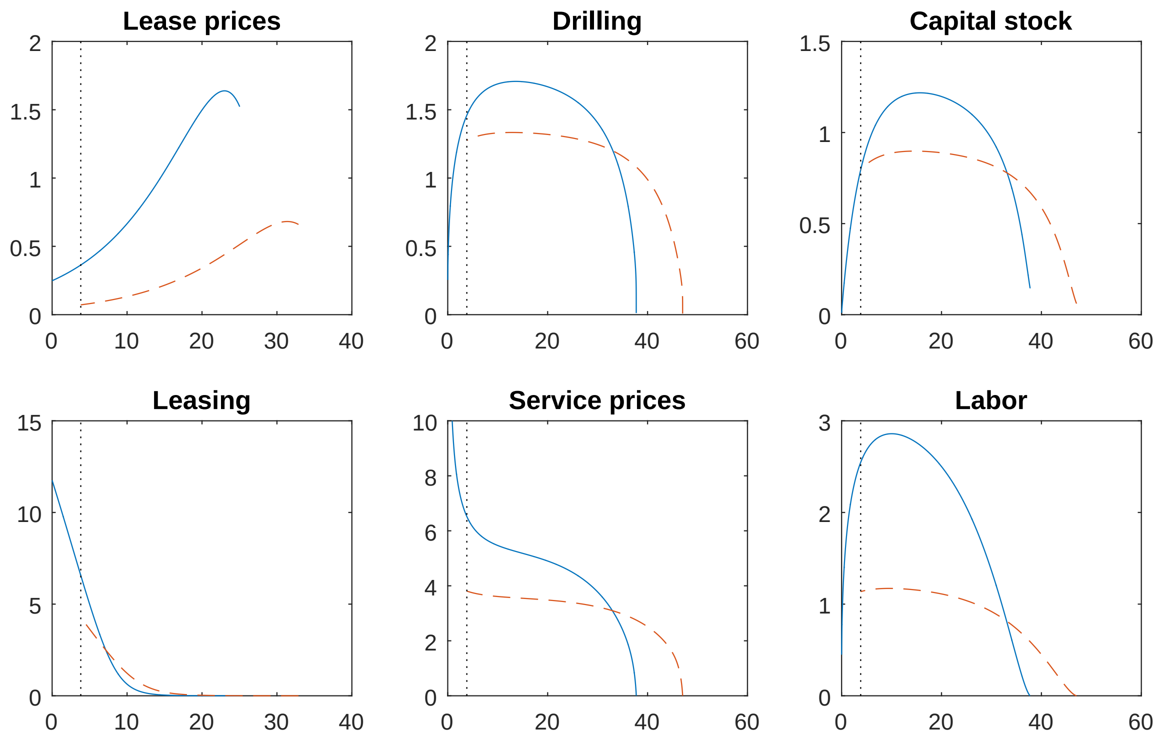


Figure 18: Simulation of unexpected 50% price drop

7 Conclusion

To my knowledge, this is the first paper in the depletable resource literature to model lease prices in a decentralized, dynamic equilibrium framework. The mineral rights market is modeled as a gold rush: mineral rights give firms the right to a share of resource rents, and because searching for mineral leases is costly and time-consuming, firms race to search early on while it is less costly to do so. This means that the empirical drilling delay that we have seen in shale plays might be thought of as an acceleration of leasing rather than a delay in drilling. This explanation contrasts with a real options explanation that requires stochastic prices to generate such a delay. The model also explains how differences in operators' drilling productivity naturally drive differences in leasing rates and, hence, the sizes of lease portfolios. Lowering barriers to entry increases competition, driving up lease prices and hastening the exhaustion of leases and drilling.

Many papers studying mineral leases focus on asymmetric information. I choose to focus on a different aspect of landowners' problem: the dynamic tradeoff between a firm's current offer with a future one at an unknown date, which may or may not come. Lease prices are determined by the future paths of firms' valuations and the arrival rates of firms to landowners. At the beginning of the shale play, landowners have many future opportunities to lease their minerals in the future and can command a larger share of the resource rent. However, they know that leasing will slow and finally stop at some point. As this happens, they have less leverage. Because firms' values for leases incorporate the entire future path of their drilling rates, costs, and technology, lease prices do as well. This provides a theoretical justification for why it may be rational for landowners to lease their minerals at the beginning of a boom instead of waiting for higher prices in the future. Though the model prediction that lease prices drop as the leasing period comes to a close, it is worth pointing out that we have by no means seen the very end of leasing the shale boom in any of the U.S. plays, so we have no data on end-of-shale lease prices, either.

A secondary contribution of this paper is to modify the classic Hotelling (1931) model in a such a way that it can match and explain more of the qualitative dynamics of the oil and gas industry that we have seen empirically. As Anderson, Kellogg, and Salant (2014) point out, the Hotelling model has lost favor in recent years because its predictions do not seem to square with the empirical reality. However, as this paper shows, we can augment the Hotelling model with learning and an oilfield services sector that uses long-lived capital to do this. At the beginning of a shale boom, a competitive oilfield services sector starts accumulating long-lived capital to reduce the price of its services. An increasing capital–labor ratio translates into lower short-run marginal costs and lower prices for oilfield services, allowing drilling to accelerate and be sustained higher levels of drilling for longer. When I simulate a 50% drop in the price of the resource to mimic the 2014 drop in the price of oil, though drilling does drop, we also see that landowners and oilfield services firms share in the pain. In both the simulation and empirical data, in fact, we see that labor and prices in oilfield services drop sharply. This allows operators to continue drilling at a moderately lower level instead of an even lower level.

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A Equilibrium equations

First, we must compute the following quantities as functions of state-variables only:

$$
\begin{aligned}
p_c: &\quad L_c^{-1}\left(\frac{p_c}{p_l}, K\right) = n\int c\left(c_w^{-1}\left(\frac{\widehat{p}(j)+\psi_w}{p_c}, A, W, \mathcal{W}; j\right), A, W, \mathcal{W}; j\right) dF(j) \\
c^s: &\quad c^s = L_c^{-1}\left(\frac{p_c}{p_l}, K\right) \\
S(j): &\quad S(j) = \psi_a(j) - V^u \\
\bar{S}: &\quad \bar{S} = \int \max\{S(j), 0\} dF(j) \\
\mathcal{V}: &\quad \mathcal{V} = \begin{cases} n\int \kappa'^{-1}\left((1-\tau)\sigma q\left(\theta\right)\max\{S(j),0\}\right) dF(j) & \text{if } \mathcal{U} > \underline{u} \\ 0 & \text{otherwise} \end{cases} \\
\mathcal{U}: &\quad \mathcal{U} = \overline{\mathcal{A}} - \mathcal{A} = \overline{\mathcal{A}} - n\int A(j) dF(j). \\
\mathcal{W}: &\quad \mathcal{W} = \int W(j) dF(j) \\
\theta: &\quad \theta = \mathcal{V}/\mathcal{U} \\
v(j): &\quad v(j) = \begin{cases} \kappa'^{-1}\left((1-\tau)\sigma q(\theta)\max\{S(j),0\}\right) & \text{if } \mathcal{U} > \underline{u} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

Then we write a system of ODEs that characterize the equilibrium for each time t in terms of state variables and the quantities calculated above:

$$
\begin{aligned}
\dot{K}: &\quad \dot{K} = p_k^{-1}(\psi_k) - \delta_k K \\
\dot{W}(j): &\quad \dot{W}(j) = w(j) = c_w^{-1}\left(\frac{\widehat{p}(j)+\psi_w}{p_c}, A, W, \mathcal{W}; j\right) \\
\dot{\psi}_k(j): &\quad \dot{\psi}_k = p_l L_k(c^s, K) + (\rho+\delta_k)\psi_k \\
\dot{\psi}_a(j): &\quad \dot{\psi}_a(j) = p_c c_A(w, A, W, \mathcal{W}; j) + \rho\psi_a \\
\dot{\psi}_w(j): &\quad \dot{\psi}_w(j) = p_c c_W(w, A, W, \mathcal{W}; j) + \rho\psi_w \\
\dot{V}^u: &\quad \dot{V}^u = \begin{cases} -\tau\theta q(\theta)\bar{S} + \rho V^u & \text{if } \mathcal{U} > \underline{u} \\ \rho V^u & \text{otherwise} \end{cases} \\
\dot{A}(j): &\quad \dot{A}(j) = v\sigma q(\theta)
\end{aligned}
$$

Recall that T is defined as as the last time where any economic activity occurs: $T = \max_{j\in J}\{T_j\}$. Then the dynamic equilibrium is further characterized by the following set of transversality conditions:

$$\begin{aligned}
V^u(T) &= 0 \\
\psi_k(T) &= p_k(-K_T) \\
\psi_a(T_j;j) &= -\psi_w(T_j;j) = \max\left\{0,\ \widehat{p}(j) - p_c(T_j)c_w\left(0, A(T_j;j), A(T_j;j), \mathcal{W}(T_j);j\right)\right\} \\
\mathcal{U}(T) &= \underline{u} \\
A(0;j) &= A_{0j} \\
W(0;j) &= W_{0j} \\
K(0;j) &= K_0 \\
\mathcal{U}(0) &= \overline{\mathcal{A}} - \int A(0;j)dF(j) \\
\mathcal{W}(0) &= \int W(0;j)dF(j)
\end{aligned}$$

As mentioned earlier, production is easily computed as

$$Q(t;j) = e^{-\delta_q t}\int_0^t e^{\delta_q \tau} q_0 w(\tau;j)d\tau + Q_{0j}e^{-\delta_q t}.$$

Together, this system of equations implies a set of stopping times $\{T_j\}_{j\in J}$ for each firm after which $w(t;j) = 0$ and a time T_0 when unleased acreage drops to the threshold where matching is not possible because $\mathcal{U}(t) = \underline{u}\forall t \geq T_0$. Finally, we require a mass of firms n such that the free entry condition, equation 25, is satisfied.

A.1 Parameterization for simulation

This also means that we can write equilibrium match-rates for firms and landowners when $\mathcal{U} > \underline{u}$ as

$$q(\theta) = \overline{m}\left[\left(\frac{\overline{m}(1-\tau)}{\kappa}\right)\frac{\sigma\bar{S}}{\mathcal{U}/n}\right]^{-\frac{1-\mu}{2}} \qquad \theta q(\theta) = \overline{m}\left[\left(\frac{\overline{m}(1-\tau)}{\kappa}\right)\frac{\sigma\bar{S}}{\mathcal{U}/n}\right]^{\mu}.$$

We can use landowners' match-rate to write the equilibrium law of motion for landowners' outside values as

$$\dot{V}^u = -\tau\overline{m}^{1+\mu}\left[\frac{1-\tau}{\kappa}\left(\frac{\sigma\bar{S}}{\mathcal{U}/n}\right)\right]^{\mu}\bar{S} + \rho V^u. \tag{53}$$

B Mathematical appendix

B.1 Derivation of landowners' transitions

One can derive equation (13) via dynamic programming by taking the limit of a discrete-time problem where the interval between periods shrinks to zero ($\epsilon \to 0$):

$$\begin{aligned} V^u(t) &= \overbrace{[1-\epsilon\theta q(\theta)]}^{\text{Pr(unmatched)}} \left[0 + \frac{V^u(t+\epsilon)}{1+\rho\epsilon}\right] + \overbrace{\epsilon\theta q(\theta)}^{\text{Pr(matched)}} \int p_a dF(j) \\ V^u(t)(1+\rho\epsilon) &= \epsilon\theta q(\theta) \int [(1+\rho\epsilon)p_a(j)]dF(j) + [1-\epsilon\theta q(\theta)]V^u(t+\epsilon) \\ [V^u(t) - V^u(t+\epsilon)] + \rho\epsilon V^u(t) &= \epsilon\theta q(\theta)\left[\int p_a(j)dF(j) - V^u(t+\epsilon)\right] \end{aligned}$$

Divide by ϵ and then cancel:

$$\begin{aligned} \frac{-[V^u(t+\epsilon) - V^u(t)]}{\epsilon} + \frac{\rho\epsilon V^u(t)}{\epsilon} &= \int [p_a - V^u(t+\epsilon)]\, dF(j) \frac{\epsilon\theta q(\theta)}{\epsilon} \\ -\frac{[V^u(t+\epsilon) - V^u(t)]}{\epsilon} + \rho V^u(t) &= \theta q(\theta) \int [p_a - V^u(t+\epsilon)]\, dF(j) \end{aligned}$$

And take a limit as $\epsilon \to 0$

$$\begin{aligned} -\dot{V}^u + \rho V^u &= \theta q(\theta)[p_a - V^u] \\ \dot{V}^u &= -\theta q(\theta) \int p_a(j)dF(j) + \rho V^u \end{aligned}$$

B.2 Original-operators-problem

The current-value Hamiltonian corresponding to operators' dynamic optimization problem, equation (17), is

$$\begin{aligned} \widetilde{\mathcal{H}} = p_q Q - p_c c(w, A, W, \mathcal{W}; j) - \kappa(v) - p_a \dot{A} \\ + \psi_a v\sigma q(\theta) + \psi_w w + \psi_q(q_0(j)w - \delta_q Q) \end{aligned}$$

After substituting in equation (16) for the flow of leases, $\dot{A}$, we can include constraints and write the Hamiltonian as

$$\begin{aligned} \widetilde{\mathcal{H}} = p_q Q - p_c c(w, A, W, \mathcal{W}; j) - \kappa(v) + (\psi_a - p_a)v\sigma q(\theta) \\ + \psi_w w + \psi_q(q_0(j)w - \delta_q Q), \end{aligned}$$

and the Lagrangian as

$$\widetilde{\mathcal{L}} = \widetilde{\mathcal{H}} + \lambda(A - W) + \lambda_w w + \lambda_v v.$$

The first-order conditions for the optimal paths of searching and drilling are

$$\frac{\partial \mathcal{L}}{\partial w} = -p_c c_w(w, A, W, \mathcal{W}; j) + \psi_w + \psi_q q_0 + \lambda_w = 0$$
$$\frac{\partial \mathcal{L}}{\partial v} = -\kappa'(v) + (\psi_a - p_a)\sigma q(\theta) + \lambda_v = 0,$$

and the co-state equations are

$$\dot{\psi}_w = p_c c_W(w, A, W, \mathcal{W}; j) + \lambda + \rho\psi_w$$
$$\dot{\psi}_a = p_c c_A(w, A, W, \mathcal{W}; j) - \lambda + \rho\psi_a$$
$$\dot{\psi}_q = -p_q + (\rho + \delta_q)\psi_q.$$

The transition equation for ψ_q only involves constants, so we can solve for a general solution:

$$\psi_q(t) = \frac{p_q}{\rho + \delta_q} + e^{(\rho+\delta_q)t}\left(\psi_q(0) - \frac{p_q}{\rho + \delta_q}\right)$$

The only stable solution for ψ_q is when $\psi_q(0) = \psi_q(t) = \frac{p_q}{\rho+\delta_q}$. This means that ψ_q must be equal to the present value of revenues from a well:

$$\psi_q(t) = \frac{p_q}{\rho + \delta_q}\forall t. \tag{54}$$

The present value of revenues from the initial stock of producing wells will simply be

$$\frac{Q_{0j} p_q}{\rho + \delta_q},$$

and the present value of expected revenues from each new well will be

$$\widehat{p}(j) \equiv \frac{p_q q_0(j)}{\rho + \delta_q}.$$

Substituting ψ_q into the FOC for drilling, w, implies that the present value of revenues from a well must equal the marginal extraction cost (MEC) it plus the opportunity cost of drilling it, also known as the marginal user cost (MUC):

$$\widehat{p}(j) = \underbrace{p_c c_w(w, A, W, \mathcal{W}; j)}_{\text{MEC}} + \underbrace{(-\psi_w)}_{\text{MUC}}.$$

We can then re-form the original problem as

$$\widetilde{V}(A_{0j}, H_{0j}, Q_{0j}; j) = \frac{p_q Q_{0j}}{\rho + \delta_q} +$$

$$\max_{w(t),v(t),T_j} \int_0^{T_j} e^{-\rho t} \left\{ \widehat{p}(j) w - p_c c(w, A, W, \mathcal{W}; j) - \kappa(v) - p_a(V^u, V^m(j), \psi_a)\dot{A} \right\} dt.$$

The solution is unique, and its necessary and sufficient conditions coincide exactly with the original problem. Thus, the two are equivalent. Finally, with an optimal path for w in hand, we can then solve for $Q(t; j)$ by integrating forward using the optimal path for w. Recall that $\dot{Q}(t; j) = q_0 w(t; j) - \delta_q Q(t; j)$. Given Q_{0j}, the solution is

$$Q(t; j) = e^{-\delta_q t} \int_0^t e^{\delta_q \tau} q_0 w(\tau; j) d\tau + Q_{0j} e^{-\delta_q t}.$$

B.3 Proofs

Lemma 5.1. *Given any admissible set of states and co-states at time t, we can find a price $p_c(t)$ to clear the drilling services market, and $p_c(t)$ is a continuous function of t.*

Proof. Equation (42) implies that

$$\frac{\widehat{p}(j) + \psi_w}{p_c} \leq c_w(w, A, W, \mathcal{W}; j),$$

and the equation holds with equality if $w > 0$. Assumption 5.3 implies that the short-run marginal services cost of drilling is strictly increasing and differentiable so the function $c_w^{-1}(\cdot)$ is well-defined, differentiable, non-negative, and weakly increasing in its first argument:

$$w(j) = c_w^{-1} \left(\frac{\widehat{p}(j) + \psi_w}{p_c}, A, W, \mathcal{W}; j \right)$$

Individual firm demand for drilling-services as a function of prices and state-variables is therefore also well-defined and differentiable. It takes non-negative values and weakly decreases in p_c:

$$c^d(j) = c \left(c_w^{-1} \left(\frac{\widehat{p}(j) + \psi_w}{p_c}, A, W, \mathcal{W}; j \right), A, W, \mathcal{W}; j \right).$$

Equation (29) implies that aggregate demand for drilling services will be another differentiable, weakly decreasing function in p_c that takes on non-negative values:

$$c^d = n \int c \left(j, c_w^{-1} \left(j, \frac{\widehat{p}(j) + \psi_w}{p_c}, A, W, \mathcal{W} \right), A, W, \mathcal{W} \right) dF(j).$$

Equation (45) implies that we can define a services supply function as

$$c^s = L_c^{-1}\left(\frac{p_c}{p_l}, K\right).$$

This continuously differentiable function is well-defined for all $(p_c, K) \in R_+^2$ and takes values in $[0, \infty)$. It is strictly increasing in both arguments. Additionally, $L_c^{-1}(0, K) = 0$ and $\lim_{x\to\infty} L_c^{-1}(x, K) = \infty$.

The equilibrium price of drilling services, p_c must satisfy the drilling services market-clearing condition:

$$L_c^{-1}\left(\frac{p_c}{p_l}, K\right) = n \int c\left(c_w^{-1}\left(\frac{\widehat{p}(j) + \psi_w}{p_c}, A, W, \mathcal{W}; j\right), A, W, \mathcal{W}; j\right) dF(j)$$

The left-hand side is strictly increasing in $p_c \in [0, \infty)$ and has range $[0, \infty)$. The right-hand side is weakly decreasing and takes non-negative values. Thus, we can define an equilibrium services price function that maps all possible admissible states, e.g., $\{\{A(t;j), W(t;j), \psi_w(t;j)\}_{j\in J}, \mathcal{A}(t), \mathcal{W}(t),$ such that $A(t;j) - W(t;j) \geq 0 \forall j, t$, into a unique market-clearing price exists. Moreover, the equilibrium services prices are a differentiable function of t by the implicit function theorem. Since the states and co-states are all continuous function of time, $p_c(t)$ is, too. □

Lemma 5.3. *Given the set of match surpluses for all types, $\{S(t;j)\}_{j\in J}$ and aggregate leasing, $\mathcal{U}(t)$, there exists a unique quantity of aggregate-searching $\mathcal{V}(t)$ that satisfies operators' optimality and the definitions of aggregate match-flows.*

Proof. First, use the FOC for searching, equation (43), to define a firm's optimal searching function as

$$v = \begin{cases} \kappa'^{-1}\left((1-\tau)\sigma q(\theta) \max\{S(j), 0\}\right) & \text{if } \mathcal{U} > \underline{u} \\ 0 & \text{otherwise} \end{cases}.$$

Assumption 5.2 implies that $\kappa^{-1}(\cdot)$ is a continuously differentiable function of $(\mathcal{U}, \mathcal{V}) \in (\underline{u}, \infty) \times (0, \infty)$ Equation (4) implies that aggregate searching when $\mathcal{U} > \underline{u}$ will be

$$\mathcal{V} = n \int \kappa'^{-1}\left((1-\tau)\sigma q\left(\frac{\mathcal{V}}{\mathcal{U}}\right) \max\{S(j), 0\}\right) dF(j).$$

The LHS is strictly increasing and continuously differentiable over the domain $\mathcal{V} \in (0, \infty)$, and the RHS is strictly decreasing and continuously differentiable when $\mathcal{V} \in (0, \infty)$. Then we can find a unique $\mathcal{V} \in (0, \infty)$ for all $S(t;j)_{j\in J}$ whenever $\mathcal{U} > \underline{u}$. We only have $\mathcal{V} = 0$ in the special case where $\mathcal{U} = \underline{u}$. □

Lemma 6.1. *Drilling is strictly declining before depletion: $\exists T_j^d < T_j$ such that $\dot{w}(t;j) < 0 \forall t > T_j^d$. Also, resource rents are strictly positive before depletion: $\exists T_j^r < T_j$ such that $-\psi_w(t;j) > 0 \forall t \in (T_j^r, T_j)$.*

Proof. The first statement follows immediately from the definition of T_j as the terminal drilling time and the continuity of $p_c(t)$, $\psi_w(t;j)$, and $N(t;j)$. For the second statement, first observe that we cannot have that $\psi_w(t;j) > 0 \forall (T_j^r, T_j)$ since this would mean rents are negative, and the firm could have done better by not drilling. Now assume for the sake of contradiction that $\psi_w(t;j) = 0 \forall t \in (T_j^r, T_j]$. This implies that $\dot{\psi}_w = c_W \forall t \in (T_j^r, T_j]$. That $-\psi_w(T_j;j) = 0$ implies we have economic depletion, so $c_W > 0 \implies \dot{\psi}_w > 0$. Thus, it must be that $\psi_w(t;j) > 0$ for some positive measure of time after T^r. Contradiction. □

Lemma 6.2. *Suppose that we have no capital, a unit-mass of homogeneous firms, and physical depletion:* $p_c = n = G(A, W) = 1$. *Now consider two firms that differ only in whether technological progress is through internalized learning-by-doing or industry-wide knowledge spillovers. Then the firm which internalizes its costs starts drilling faster and depletes its resource first.*

We proceed in two steps. First, we show by contradiction that the firm with private learning cannot finish at the same time or after the firm with spillovers. Then, we show that the private learning firm's co-state crosses the spillover-firm's co-state once from above. That the private firm's drilling at $t = 0$ follows immediately.

Proof. Definitions. Denote the cost-function for the internalizing firm as $\boldsymbol{c(w, W)} = E(w)D(W)$, and the second, as $c(w, \mathcal{W}) = E(w)\mathcal{D}(\mathcal{W})$. Denote depletion time with private learning-by-doing as $\boldsymbol{T}$, and T for the spillovers case. Additionally, assume that the learning functions are the same: $D(x) = \mathcal{D}(x)$. No depletion effects implies that $\boldsymbol{c_W} < 0$. We know that the firms both lease and drill the same amount, W_T, and that terminal rents are the same: $\boldsymbol{\Psi}_w(\boldsymbol{T}) = \psi_w(T) = \psi_{wT}$.

Step 1) The private-learner cannot finish second. Integrate the co-state equations:

$$\psi_w(t) = \psi_{wT} e^{-\rho(T-t)}$$
$$\boldsymbol{\Psi}_w(t) = \psi_{wT} e^{-\rho(\boldsymbol{T}-t)} + e^{\rho t} \int_t^{\boldsymbol{T}} e^{-\rho s}(-\boldsymbol{c_W}) ds.$$

Since $-\boldsymbol{c_W} > 0$, we know that for all $\Delta \in (0, \min\{\boldsymbol{T}, T\})$, the following is true:

$$\boldsymbol{\Psi}_w(\boldsymbol{T} - \Delta) > \psi_w(T - \Delta).$$

The path of cumulative drilling under spillovers must satisfy

$$W_T - W(T - \Delta) = \int_0^{\Delta} (E')^{-1} \left(\frac{\hat{p} + \psi_w(T - s)}{D(W(T - s))} \right) ds.$$

The fact that $E' > 0$ and $\boldsymbol{\Psi}_w(\boldsymbol{T} - s) > \psi_w(T - s)$ implies that the following inequality must hold for $\Delta \in (0, \min\{\boldsymbol{T}, T\})$:

$$\int_0^\Delta (E')^{-1}\left(\frac{\hat{p}+\psi_w(T-s)}{D(W(T-s))}\right) ds < \int_0^\Delta (E')^{-1}\left(\frac{\hat{p}+\boldsymbol{\Psi}_w(\boldsymbol{T}-s)}{D(W(T-s))}\right) ds.$$

Differentiate both sides with respect to Δ:

$$\dot{W}(T-\Delta) = (E')^{-1}\left(\frac{\hat{p}+\psi_w(T-\Delta)}{D(W(T-\Delta))}\right) < (E')^{-1}\left(\frac{\hat{p}+\boldsymbol{\Psi}_w(\boldsymbol{T}-\Delta)}{D(W(T-\Delta))}\right)$$

Define $t = T - \Delta$, which implies that $\boldsymbol{T} - \Delta = t + [\boldsymbol{T} - T]$ and assume for the sake of contradiction that depletion under private learning is after depletion under spillovers: $\boldsymbol{T} \geq T$. Then we can see that

$$\dot{W}(t) = (E')^{-1}\left(\frac{\hat{p}+\psi_w(t)}{D(W(t))}\right) < (E')^{-1}\left(\frac{\hat{p}+\boldsymbol{\Psi}_w(t+[\boldsymbol{T}-T])}{D(W(t))}\right) \leq (E')^{-1}\left(\frac{\hat{p}+\boldsymbol{\Psi}_w(t)}{D(W(t))}\right),$$

where the second inequality follows from $\dot{\psi}_w < 0$ and $\dot{\boldsymbol{\Psi}}_w < 0$. Thus, we know that $\hat{p} + \psi_w(t) < \hat{p} + \boldsymbol{\Psi}_w(t) \forall t$. Furthermore, setting $t = 0$ implies that initial drilling with private learning is larger than initial drilling with public spillovers:

$$\dot{W}(0) = (E')^{-1}\left(\frac{\hat{p}+\psi_w(0)}{D(W_0)}\right) < (E')^{-1}\left(\frac{\hat{p}+\boldsymbol{\Psi}_w([\boldsymbol{T}-T])}{D(W_0)}\right) \leq (E')^{-1}\left(\frac{\hat{p}+\boldsymbol{\Psi}_w(0)}{D(W_0)}\right) = \dot{\boldsymbol{W}}(0).$$

Continuity implies that $\exists \epsilon > 0$ such that $\forall t \in (0, \epsilon)$, we have $\boldsymbol{W}(t) > W(t)$. However, $\boldsymbol{T} > T$ is true if and only if $\exists T^* > \epsilon$ such that $W(t) \geq \boldsymbol{W}(t) \forall t \geq T^*$. For such an over-taking to occur, we must have that $\psi_w(t) > \boldsymbol{\Psi}_w(t)$. This contradicts the fact that $\psi_w(t) < \boldsymbol{\Psi}_w(t) \forall t$. Therefore, it must be that case that $\boldsymbol{T} < T$.

Step 2) The private-learner must start drilling faster.

Since $\boldsymbol{T} < T$, we know that $\boldsymbol{\Psi}_w(\boldsymbol{T}) < \psi_w(\boldsymbol{T})$. We first show that if the two cross, it will be just once. Then we show that they must. Compute the difference in co-states as

$$\begin{aligned}\psi_w(t) - \boldsymbol{\Psi}_w(t) &= \psi_{wT}\left[e^{-\rho(T-t)} - e^{-\rho(\boldsymbol{T}-t)}\right] + e^{\rho t}\int_t^{\boldsymbol{T}} e^{-\rho s} \boldsymbol{c}_{\boldsymbol{W}} ds \\ &= e^{\rho t}\left\{-\psi_{wT}\left[e^{-\rho \boldsymbol{T}} - e^{-\rho T}\right] + \int_t^{\boldsymbol{T}} e^{-\rho s} \boldsymbol{c}_{\boldsymbol{W}} ds\right\}.\end{aligned}$$

The first term inside the brackets is positive, and the last, negative. Taking a derivative with respect to time leads to

$$\begin{aligned}\dot{\psi}_w(t) - \dot{\boldsymbol{\Psi}}_w(t) &= \rho e^{\rho t}\left\{-\psi_{wT}\left[e^{-\rho \boldsymbol{T}} - e^{-\rho T}\right] + \int_t^{\boldsymbol{T}} e^{-\rho s}\boldsymbol{c_W} ds\right\} + (-\boldsymbol{c_W}) \\ &= \rho\left[\psi_w(t) - \boldsymbol{\Psi}(t)\right] + (-\boldsymbol{c_W})\end{aligned}$$

This implies that at any crossing-point where $\psi_w(t) - \boldsymbol{\Psi}_w(t) = 0$, $\dot{\psi}_w(t) - \dot{\boldsymbol{\Psi}}_w(t) = -\boldsymbol{c_W} > 0$. Since the derivative is strictly increasing at any zero-crossing, we can only have one.

Now assume for contradiction that there is no zero-crossing. Then $\boldsymbol{\Psi}_w(0) < \psi_w(0)$. This implies that $\dot{W}(0) > \dot{\boldsymbol{W}}(0)$, so $\exists \epsilon > 0$ such that $W(t) > \boldsymbol{W}(t) \forall t \in (0, \epsilon)$. For us to have that $W(\boldsymbol{T}) < \boldsymbol{W}(\boldsymbol{T})$, it must be that at some point $\boldsymbol{\Psi}_w(t) > \psi_w(t)$ so that $\boldsymbol{W}(t)$ can overtake $W(t)$. Continuity then implies a crossing point, which contradicts our assumption. Therefore, we must have a zero-crossing point, which will be unique. The fact that $\boldsymbol{\Psi}_w(0) > \psi(0)$ implies $\dot{\boldsymbol{W}}(0) > \dot{W}(0)$.

This implies that with private learning, initial drilling is faster, and depletion is sooner. □

Lemma 6.3. *For drilling unprofitable wells to be optimal, learning-by-doing must have a private component*

Proof. Suppose not. Purely public spillovers imply that $c_W \geq 0$, and unprofitable drilling implies that $\dot{\psi}_w < 0$. The co-state transition equation, equation (35), implies that $\dot{\psi}_w = p_c c_W(w, A, W, \mathcal{W}; j) + \lambda + \rho\psi_w 0$. This means that all future wells will be even more unprofitable. The firm could increase its profits by not drilling any wells, which contradicts that it was optimal to drill a positive measure of wells. □

Lemma 6.5. *Suppose that firms are differentiated in costs but not initial production rates: $\varphi(j) \neq \varphi(j') \forall j \neq j'$ and $q_0(j) = q_0(j') \forall j \neq j'$. Also suppose that there is no private learning-by-doing: $D(W) = 1$. Then the more efficient firms finish drilling faster and have a bigger lease portfolio.*

Proof. Since $q_0(j) = q_0(j') \forall j \neq j'$ and $E'(0) = 0$, we know that terminal resource rents are the same: $\psi_{aT}(j) = -\psi_{wT}(j) = \widehat{p} > 0$. All firms lease during the same period $[0, T_0]$, so the firm with a bigger portfolio values leases more:

$$A(T_0; j) > A(T_0, j') \iff \psi_a(t; j) = -\psi_w(t; j) > -\psi_w(t; j') = \psi_a(t; j').$$

Furthermore, the one with the higher value must finish drilling first:

$$\psi_a(t; j) = e^{-\rho(T_j - t)}\widehat{p} > e^{-\rho(T_{j'} - t)}\widehat{p} = \psi_a(t; j') \iff T_j < T_{j'}.$$

Physical depletion implies that $A(T_0; j) = W(T_j; j)$, and we can compute $W(t; j)$ as

$$W(t;j) = \frac{1}{\varphi(j)} \int_0^t \frac{\widehat{p} + \psi_w(s;j)}{p_c(s)\mathcal{D}(\mathcal{W}(s))} ds.$$

If firm j drills more during $[0, T_j]$, e.g., $W(T_j; j) > W(T_j; j')$, and $\psi_w(t; j) < \psi_w(t; j')$, it must be that firm j has lower costs that firm j': $\varphi(j) < \varphi(j')$.

Assume for the sake of contradiction that firm j has lower costs than firm j', $\varphi(j) < \varphi(j')$, and it leases less: $A(T_0; j) < A(T_0; j') \iff \psi_a(t; j) < \psi_a(t; j')$. This means that (1) firm j' must finish drilling before firm j

$$A(T_0, j') = \frac{1}{\varphi(j')} \int_0^{T_{j'}} \frac{\hat{p} + \psi_w(t;j')}{p_c(t)\mathcal{D}(\mathcal{W}(t))} dt > \frac{1}{\varphi(j)} \int_0^{T_{j'}} \frac{\hat{p} + \psi_w(t;j)}{p_c(t)\mathcal{D}(\mathcal{W}(t))} dt.$$

However, this contracts the fact that $\varphi(j) < \varphi(j')$ and $\psi_w(t; j') < \psi_w(t; j)$. □